CHAPTER 5

NORMED SPACES AND L_p-SPACES

27. NORMED SPACES AND BANACH SPACES

Problem 27.1. *Let X be a normed space. Then show that X is a Banach space if and only if its unit sphere $\{x \in X\colon \|x\| = 1\}$ is a complete metric space (under the induced metric $d(x, y) = \|x - y\|$).*

Solution. Let $S = \{x \in X\colon \|x\| = 1\}$, and note that S is a closed set. Clearly, if X is a Banach space, then S is a complete metric space.

Conversely, assume that S is complete. Let $\{x_n\}$ be a Cauchy sequence of X. In view of the inequality

$$\big|\, \|x_n\| - \|x_m\| \,\big| \le \|x_n - x_m\|,$$

we see that $\{\|x_n\|\}$ is a Cauchy sequence of real numbers. If $\lim \|x_n\| = 0$, then $\lim x_n = 0$. So, we can assume that $\delta = \lim \|x_n\| > 0$. In this case, we can also assume that there exists some $M > 0$ such that $\frac{1}{\|x_n\|} \le M$ and $\|x_n\| \le M$ both hold for each n. The inequalities

$$\begin{aligned}\Big\| \tfrac{x_n}{\|x_n\|} - \tfrac{x_m}{\|x_m\|} \Big\| &= \tfrac{\|(\|x_m\|x_n - \|x_n\|x_m)\|}{\|x_n\|\cdot\|x_m\|} \\ &\le M^2 \,\big\|\, \|x_m\|(x_n - x_m) - (\|x_n\| - \|x_m\|)x_m \,\big\| \\ &\le 2M^3\|x_n - x_m\|,\end{aligned}$$

show that the sequence $\Big\{\frac{x_n}{\|x_n\|}\Big\}$ is a Cauchy sequence of S. If x is its limit in S, then $x_n = \|x_n\| \cdot \frac{x_n}{\|x_n\|} \longrightarrow \delta x$ holds in X, and so X is a Banach space.

Problem 27.2. *Let X be a normed vector space. Fix $a \in X$ and a nonzero scalar α.*

a. *Show that the mappings $x \mapsto a + x$ and $x \mapsto \alpha x$ are both homeomorphisms.*
b. *If A and B are two sets with either A or B open and α and β are nonzero scalars, then show that $\alpha A + \beta B$ is an open set.*

Solution. (a) Observe that $\|(a + x) - (a + y)\| = \|x - y\|$ holds for all x and y. This shows that $x \mapsto a + x$ is, in fact, an isometry.

Also notice that for all $x, y \in X$ we have $\|\alpha x - \alpha y\| = |\alpha| \cdot \|x - y\|$. This easily implies that $x \mapsto \alpha x$ is a homeomorphism.

(b) Assume first that B is an open set. Since the mapping $x \mapsto a + x$ is a homeomorphism, we know that $a + B$ is an open set for each $a \in X$. This implies that the set $A + B = \bigcup_{a \in A}(a + B)$ is an open set for each subset A of X.

Now, assume that B is an open set and that α and β are nonzero scalars. Since the mapping $x \mapsto \beta x$ is a homeomorphism, the set βB is an open set. So, by the preceding discussion, $\alpha A + \beta B$ must be an open set.

Problem 27.3. *Let X be a normed vector space, and let $B = \{x \in X\colon \|x\| < 1\}$ be its open unit ball. Show that $\overline{B} = \{x \in X\colon \|x\| \le 1\}$.*

Solution. Repeat the solution of Problem 6.2.

Problem 27.4. *Let X be a normed space, and let $\{x_n\}$ be a sequence of X such that $\lim x_n = x$ holds. If $y_n = n^{-1}(x_1 + \cdots + x_n)$ for each n, then show that $\lim y_n = x$.*

Solution. Let $\varepsilon > 0$. Choose some k with $\|x_n - x\| < \varepsilon$ for all $n > k$. Fix some $m > k$ so that $\|\frac{1}{n}[(x_1 - x) + \cdots + (x_k - x)]\| < \varepsilon$ holds for all $n > m$. Thus, if $n > m$, then

$$\begin{aligned}\|y_n - x\| &= \left\|\tfrac{1}{n}\left[(x_1 - x) + \cdots + (x_k - x) + (x_{k+1} - x) + \cdots + (x_n - x)\right]\right\| \\ &\le \left\|\tfrac{1}{n}\left[(x_1 - x) + \cdots + (x_k - x)\right]\right\| + \tfrac{1}{n}\left[\|x_{k+1} - x\| + \cdots + \|x_n - x\|\right] \\ &< \varepsilon + \varepsilon = 2\varepsilon.\end{aligned}$$

That is, $\lim y_n = x$ holds. (See also Problem 4.11.)

Problem 27.5. *Assume that two vectors x and y in a normed space satisfy $\|x + y\| = \|x\| + \|y\|$. Then show that*

$$\|\alpha x + \beta y\| = \alpha\|x\| + \beta\|y\|$$

holds for all scalars $\alpha \ge 0$ and $\beta \ge 0$.

Solution. Assume $\|x + y\| = \|x\| + \|y\|$ holds for two vectors x and y in a normed space, and let $\alpha \ge 0$ and $\beta \ge 0$. Without loss of generality, we can

suppose that $\alpha \geq \beta \geq 0$. From the triangle inequality, it follows that

$$\|\alpha x + \beta y\| \leq \alpha\|x\| + \beta\|y\|.$$

Next, notice that

$$\begin{aligned}\|\alpha x + \beta y\| &= \big\|\alpha(x+y) + (\beta - \alpha)y\big\| \\ &\geq \big\|\alpha(x+y)\big\| - \big\|(\beta-\alpha)y\big\| \\ &= \alpha\|x+y\| - (\alpha - \beta)\|y\| = \alpha\big(\|x\| + \|y\|\big) - (\alpha-\beta)\|y\| \\ &= \alpha\|x\| + \beta\|y\|.\end{aligned}$$

Hence, $\|\alpha x + \beta y\| = \alpha\|x\| + \beta\|y\|$, as desired.

Problem 27.6. *Let X be the vector space of all real-valued functions defined on $[0, 1]$ having continuous first-order derivatives. Show that $\|f\| = |f(0)| + \|f'\|_\infty$ is a norm on X that is equivalent to the norm $\|f\|_\infty + \|f'\|_\infty$.*

Solution. The verification of the norm properties of $\|\cdot\|$ are straightforward. From

$$f(x) = f(0) + \int_0^x f'(t)\,dt,$$

we see that $|f(x)| \leq |f(0)| + \|f'\|_\infty$ holds for each $x \in [0, 1]$, and consequently, $\|f\|_\infty \leq |f(0)| + \|f'\|_\infty$ holds.

The equivalence of the two norms follows from the inequalities

$$\begin{aligned}\|f\|_\infty + \|f'\|_\infty &\leq |f(0)| + 2\|f'\|_\infty \leq 2\big(|f(0)| + \|f'\|_\infty\big) \\ &= 2\|f\| \leq 2\big(\|f\|_\infty + \|f'\|_\infty\big).\end{aligned}$$

Problem 27.7. *A series $\sum_{n=1}^\infty x_n$ in a normed space is said to* **converge** *to x if $\lim \|x - \sum_{i=1}^n x_i\| = 0$. As usual, we write $x = \sum_{n=1}^\infty x_n$. A series $\sum_{n=1}^\infty x_n$ is said to be* **absolutely summable** *if $\sum_{n=1}^\infty \|x_n\| < \infty$ holds.*

Show that a normed space X is a Banach space if and only if every absolutely summable series is convergent.

Solution. Let X be a Banach space, and let $\sum_{n=1}^\infty x_n$ be an absolutely summable series. For each n let $s_n = \sum_{i=1}^n x_i$. The inequality

$$\big\|s_{n+p} - s_n\big\| = \Big\|\sum_{i=n+1}^{n+p} x_i\Big\| \leq \sum_{i=n+1}^{n+p} \|x_i\|$$

implies that $\{s_n\}$ is a Cauchy sequence, and hence, convergent in X.

For the converse, let $\{x_n\}$ be a Cauchy sequence in a normed space X whose absolutely summable series are convergent. By passing to a subsequence (if necessary), we can assume that $\|x_{n+1}-x_n\| < 2^{-n}$ holds for each n. Put $x_0 = 0$, and note that $\sum_{n=0}^{\infty}\|x_{n+1}-x_n\| < \infty$. Thus, $\lim_{n\to\infty}\sum_{i=0}^{n-1}(x_{i+1}-x_i) = \lim_{n\to\infty} x_n$ exists in X so that X is a Banach space.

Problem 27.8. *Show that a closed proper vector subspace of a normed vector space is nowhere dense.*

Solution. Let E be a proper closed subspace of a normed space X. Assume that E has an interior point a. Then, there exists some $r > 0$ such that $B(a,r) \subseteq E$. Now, if y is an arbitrary nonzero element of X, then $a + \frac{r}{2\|y\|}y \in B(a,r) \subseteq E$, and so $y = \frac{2\|y\|}{r}\big[(a+\frac{r}{2\|y\|}y) - a\big] \in E$. That is, $E = X$ holds, a contradiction. Thus, $E^\circ = \emptyset$.

Problem 27.9. *Assume that $f\colon [0,1] \to \mathbf{R}$ is a continuous function which is not a polynomial. By Corollary 11.6 we know that there exists a sequence of polynomials $\{p_n\}$ that converges uniformly to f. Show that the set of natural numbers*

$$\{k \in \mathbf{N}\colon\ k = \text{degree of } p_n \text{ for some } n\}$$

is countable.

Solution. Let $f\colon [0,1] \to \mathbf{R}$ be a continuous function which is not a polynomial, and let $\{p_n\}$ be a sequence of polynomials that converges uniformly to f on $[0,1]$. Assume by way of contradiction that the set of natural numbers

$$K = \{k \in \mathbf{N}\colon\ k = \text{degree of } p_n \text{ for some } n\}$$

is bounded. This means that there exists some $m \in \mathbf{N}$ such that every p_n has degree at-most m. So, if V is the finite dimensional vector subspace generated in $C[0,1]$ by the functions $\{1, x, x^2, \ldots, x^m\}$, then $\{p_n\} \subseteq V$ holds. Now, a glance at Theorem 27.7 guarantees that V is a closed subspace of $C[0,1]$, and thus the (sup) norm limit f of $\{p_n\}$ must lie in V. That is, f must be a polynomial of degree at-most m, contrary to our hypothesis. Hence, K is not bounded, and therefore it must be a countable set; see Theorem 2.4.

Problem 27.10. *This problem describes some important classes of subsets of a vector space. A nonempty subset A of a vector space X is said to be:*

a. **symmetric,** *if $x \in A$ implies $-x \in A$, i.e., if $A = -A$;*

b. **convex**, *if* $x, y \in A$ *implies* $\lambda x + (1 - \lambda)y \in A$ *for all* $0 \leq \lambda \leq 1$, *i.e., for every two vectors* $x, y \in A$ *the line segment joining* x *and* y *lies in* A; *and*

c. **circled** (*or* **balanced**) *if* $x \in A$ *implies* $\lambda x \in A$ *for each* $|\lambda| \leq 1$.

Establish the following:

i. *A circled set is symmetric.*
ii. *A convex and symmetric set containing zero is circled.*
iii. *A nonempty subset* A *of a vector space is convex if and only if* $aA + bA = (a + b)A$ *holds for all scalars* $a \geq 0$ *and* $b \geq 0$.
iv. *If* A *is a convex subset of a normed space, then the closure* $\overline{A}$ *and the interior* A° *of* A *are also convex sets.*

Solution. (i) Let A be a circled set. Since $|-1| = 1 \leq 1$, it follows that $-x = (-1)x \in A$ for each $x \in A$. Thus, A is a symmetric set.

(ii) Let A be a convex symmetric set containing zero. Fix $x \in A$ and $|\lambda| \leq 1$. If $0 \leq \lambda \leq 1$, then $\lambda x = \lambda x + (1 - \lambda)0 \in A$ and if $-1 \leq \lambda < 0$, then $\lambda x = (-\lambda)(-x) + (1 + \lambda)0 \in A$. So, A is a circled set.

(iii) Let A be a subset of a vector space. Assume first that A is a convex set, and let $a > 0$ and $b > 0$. If $x \in (a + b)A = \{(a + b)u\colon u \in A\}$, then for some $u \in A$, we have $x = (a + b)u = au + bu \in aA + bA$, and so $(a + b)A \subseteq aA + bA$ is always true. Now, let $x \in aA + bA$. Then, there exist $u, v \in A$ such that $x = au + bv$. Since A is convex, we have $z = \frac{a}{a+b}u + \frac{b}{a+b}v \in A$, and so $x = (a+b)\left[\frac{a}{a+b}u + \frac{b}{a+b}v\right] = (a+b)z \in (a+b)A$. Therefore, $aA+bA \subseteq (a+b)A$ is also true, and consequently $aA + bA = (a + b)A$.

Next, suppose that $aA + bA = (a + b)A$ holds true for all $a \geq 0$ and $b \geq 0$. Let $x, y \in A$ and $0 \leq \lambda \leq 1$. Letting $a = \lambda$ and $b = 1 - \lambda$, we see that

$$\lambda x + (1 - \lambda)y = ax + by \in (a + b)A = A.$$

This shows that A is a convex set.

(iv) Let A be a convex subset of a normed space. We show first that $\overline{A}$ is a convex set. To this end, let $x, y \in \overline{A}$ and fix $0 \leq \lambda \leq 1$. Pick two sequences $\{x_n\}$ and $\{y_n\}$ of A such that $x_n \to x$ and $y_n \to y$. Put $z_n = \lambda x_n + (1 - \lambda)y_n$ and note that $\{z_n\}$ is a sequence of A. Since the function $f\colon X \to X$, defined by $f(u) = \lambda u + (1 - \lambda)u$, is continuous (see Problem 27.2), it follows that $z_n \to \lambda x + (1 - \lambda)y$. This implies $\lambda x + (1 - \lambda)y \in \overline{A}$, so that $\overline{A}$ is a convex set.

Next, we shall show that A° is a convex set. Fix $0 \leq \lambda \leq 1$. Since A° is an open set, it follows (from Problem 27.2) that the set $\lambda A^{\circ} + (1 - \lambda)A^{\circ}$ is also an open set, which (since A is convex) is contained in A. Since A° is the largest open set contained in A, we infer that $\lambda A^{\circ} + (1 - \lambda)A^{\circ} \subseteq A^{\circ}$. This shows that A° is a convex set.

Problem 27.11. *This problem describes all norms on a vector space X that are equivalent to a given norm. So, let $(X, \|\cdot\|)$ be a normed vector space. Let A be a norm bounded convex symmetric subset of X having zero as an interior point (relative to the topology generated by the norm $\|\cdot\|$). Define the function $p_A: X \to \mathbb{R}$ by*

$$p_A(x) = \inf\{\lambda > 0\colon\ x \in \lambda A\}.$$

Establish the following:

a. *The function p_A is a well-defined norm on X.*
b. *The norm p_A is equivalent to $\|\cdot\|$, i.e., there exist two constants $C > 0$ and $K > 0$ such that $C\|x\| \le p_A(x) \le K\|x\|$ holds for all $x \in X$.*
c. *The closed unit ball of p_A is the closure of A, i.e., $\{x \in X\colon\ p_A(x) \le 1\} = \overline{A}$.*
d. *Let $|||\cdot|||$ be a norm on X which is equivalent to $\|\cdot\|$, and consider the norm bounded nonempty symmetric convex set $B = \{x \in X\colon\ |||x||| \le 1\}$. Then zero is an interior point of B and $|||x||| = p_B(x)$ holds for each $x \in X$.*

Solution. Assume that A is a norm bounded convex symmetric subset of a normed space $(X, \|\cdot\|)$ such that zero is an interior point of A.

(a) Pick some $r > 0$ such that $B(0, 2r) = \{x \in X\colon\ \|x\| < 2r\} \subseteq A$. If $x \in X$ is a nonzero vector, then $\frac{r}{\|x\|}x \in B(0, 2r) \subseteq A$, and so $x \in \frac{\|x\|}{r}A$. This shows that the set $\{\lambda > 0\colon\ x \in \lambda A\}$ is nonempty and so the formula $p_A(x) = \inf\{\lambda > 0\colon\ x \in \lambda A\}$ is well defined and satisfies

$$p_A(x) \le r\|x\| \qquad (\star)$$

for all $x \in X$. Next, we shall show that p_A is a norm on X.

Clearly, $p_A(x) \ge 0$ and $p_A(0) = 0$. Now, if $p_A(x) = 0$, then there exist a sequence $\{a_n\} \subseteq A$ and a sequence $\{\lambda_n\}$ of positive real numbers satisfying $\lambda_n \to 0$ and $x = \lambda_n a_n$ for each n. Since A is a norm bounded set, it easily follows that $x = \lim \lambda_n a_n = 0$. Thus, $p_A(x) = 0$ if and only if $x = 0$.

Next, we shall show that $p_A(\alpha x) = |\alpha| p_A(x)$ holds for all $\alpha \in \mathbb{R}$ and all $x \in X$. Since A is symmetric, we have

$$\{\lambda > 0\colon\ \lambda x \in A\} = \{\lambda > 0\colon\ \lambda(-x) \in A\},$$

and so for proving $p_A(\alpha x) = |\alpha| p_A(x)$, we can suppose without loss of generality that $\alpha > 0$. Now, note that

$$\begin{aligned} p_A(\alpha x) &= \inf\{\lambda > 0\colon\ \alpha x \in \lambda A\} = \inf\{\lambda > 0\colon\ x \in \tfrac{\lambda}{\alpha} A\} \\ &= \alpha \inf\{\tfrac{\lambda}{\alpha}\colon\ x \in \tfrac{\lambda}{\alpha} A\} = \alpha \inf\{\mu > 0\colon\ x \in \mu A\} \\ &= \alpha p_A(x). \end{aligned}$$

For the triangle inequality, let $x, y \in X$ and fix $\epsilon > 0$. Choose $\lambda > 0$ and $x \in \lambda A$ such that $\lambda < p_A(x)+\epsilon$. Likewise, pick some $\mu > 0$ such that $y \in \mu A$ and $\mu < p_A(y)+\epsilon$. From Problem 27.10 we know that $x+y \in \lambda A+\mu A = (\lambda+\mu)A$, and so

$$p_A(x+y) \le \lambda + \mu < \left[p_A(x)+\epsilon\right] + \left[p_A(y)+\epsilon\right] = p_A(x) + p_A(y) + 2\epsilon.$$

Since $\epsilon > 0$ is arbitrary, we infer that $p_A(x+y) \le p_A(y) + p_A(y)$.

(b) Let $x \in X$ and fix some $M > 0$ such that $\|a\| \le M$ holds for all $a \in A$. Now, if $\lambda > 0$ satisfies $x \in \lambda A$, then there exists some $y \in A$ such that $x = \lambda y$. Hence, $\|x\| = \lambda\|y\| \le \lambda M$, or $\lambda \ge \|x\|/M$. This implies $p_A(x) \ge \frac{1}{M}\|x\|$. Now, combine this inequality with ($\star$) to establish that p_A is a norm equivalent to $\|\cdot\|$.

(c) Assume first that $p_A(x) \le 1$ and $x \ne 0$. Then for each n there exist $0 < \lambda_n \le 1+\frac{1}{n}$ and $a_n \in A$ such that $x = \lambda_n a_n$. By passing to a subsequence, we can assume $\lambda_n \to \lambda$. Since $x \ne 0$ and A is a norm bounded set, it easily follows that $0 < \lambda \le 1$. Now, note that the sequence $\{a_n\} \subseteq A$ satisfies $a_n = \frac{1}{\lambda_n}x \to \frac{1}{\lambda}x$, and so $\frac{1}{\lambda}x \in \overline{A}$. Since $\overline{A}$ is also a convex set (see Problem 27.10), we see that $x = \lambda\left(\frac{1}{\lambda}x\right) + (1-\lambda)0 \in \overline{A}$.

Now, let $x \in \overline{A}$. Then, there exists a sequence $\{x_n\} \subseteq A$ such that $\|x_n - x\| \to 0$. Since $\|\cdot\|$ is equivalent to p_A, we also have $p_A(x_n - x) \to 0$. In particular, $p_A(x_n) \to p_A(x)$. Now, notice that since $x_n \in A$, we have $p_A(x_n) \le 1$ for each n. This implies $p_A(x) \le 1$. Therefore, the closed unit ball of p_A is $\overline{A}$.

(d) Let $|||\cdot|||$ be a norm on X which is equivalent to $\|\cdot\|$. It is easy to check that the closed unit ball B of $|||\cdot|||$ is a bounded convex and symmetric set containing zero as an interior point. We shall show next that $|||x||| = p_B(x)$ holds for each $x \in X$.

To see this, let $x \in X$ be a nonzero vector. Since $x/|||x||| \in B$, we see that $p_B(x)/|||x||| = p_B\left(x/|||x|||\right) \le 1$, and so $p_B(x) \le |||x|||$. On the other hand, there exist a sequence $\{\lambda_n\}$ of positive real numbers and a sequence $\{b_n\}$ of B such that $\lambda_n \to p_B(x)$, $b_n \in B$ and $x = \lambda_n b_n$ for each n. Since $|||x||| = \lambda_n|||b_n||| \le \lambda_n$, we easily infer that $|||x||| \le p_B(x)$. Hence, $p_B(x) = |||x|||$ for each $x \in X$.

28. OPERATORS BETWEEN BANACH SPACES

Problem 28.1. *Let X and Y be two Banach spaces and let $T: X \to Y$ be a bounded linear operator. Show that either T is onto or else $T(X)$ is a meager set.*

Solution. Assume that $T(X)$ is not a meager set. Then, we have to show that $T(X) = Y$ holds.

Let $V = \{x \in X: \|x\| \le 1\}$. Since (by assumption) $T(X)$ is not a meager set, some $\overline{nT(V)} = n\overline{T(V)}$ has an interior point. This implies that $\overline{T(V)}$ has an interior point. So, there exists some $y_0 \in \overline{T(V)}$ and some $r > 0$ such that

$B(y_0, 2r) \subseteq \overline{T(V)} = -\overline{T(V)}$. Note that if $y \in Y$ satisfies $\|y\| < 2r$, then $y - y_0 = -(y_0 - y) \in \overline{T(V)}$ and so $y = (y - y_0) + y_0 \in \overline{T(V)} + \overline{T(V)} \subseteq 2\overline{T(V)}$. (The last inclusion follows, of course, from the identity $V + V = 2V$.) Consequently, we have established that $\{y \in Y\colon \|y\| < r\} \subseteq \overline{T(V)}$. From the linearity of T, we infer that

$$\{y \in Y\colon \|y\| < 2^{-n}r\} \subseteq 2^{-n}\overline{T(V)} = \overline{T(2^{-n}V)} \tag{$\star\star$}$$

holds for each n.

Next, let $y \in Y$ be fixed such that $\|y\| < 2^{-1}r = \frac{r}{2}$. From $(\star\star)$, we know that $y \in \overline{T(2^{-1}V)}$. So, for some vector $x_1 \in 2^{-1}V$ we have $\|y - T(x_1)\| < 2^{-2}r$. Now, proceed inductively. Assume that $x_n \in 2^{-n}V$ has been selected such that $\left\|y - \sum_{i=1}^{n} T(x_i)\right\| < 2^{-n-1}r$. From $(\star\star)$ it follows that $y - \sum_{i=1}^{n} T(x_i) \in \overline{T(2^{-n-1}V)}$, and so there exists some $x_{n+1} \in 2^{-n-1}V$ such that $\left\|y - \sum_{i=1}^{n+1} T(x_i)\right\| < 2^{-n-2}r$. Thus, there exists a sequence $\{x_n\}$ of X such that $\|x_n\| \le 2^{-n}$ and

$$\left\|y - \sum_{i=1}^{n} T(x_i)\right\| = \left\|y - T\Big(\sum_{i=1}^{n} x_i\Big)\right\| < 2^{-n-1}r$$

hold for each n. Now, for each n let $s_n = x_1 + \cdots + x_n$ and note that the relation

$$\|s_{n+p} - s_n\| = \left\|\sum_{i=n+1}^{n+p} x_i\right\| \le \sum_{i=n+1}^{n+p} \|x_i\| \le \sum_{i=n+1}^{\infty} 2^{-i} = 2^{-n}$$

shows that $\{s_n\}$ is a Cauchy sequence of X. Since X is a Banach space, the sequence $\{s_n\}$ converges; let $s = \lim s_n$. Clearly, $\|s\| \le \sum_{n=1}^{\infty} \|x_n\| \le 1$ (i.e., $s \in V$), and by the continuity and linearity of T, we see that

$$T(s) = \lim_{n\to\infty} T(s_n) = \lim_{n\to\infty} \sum_{i=1}^{n} T(x_i) = y.$$

That is, $y \in T(V)$, and so $\{y \in Y\colon \|y\| < \frac{r}{2}\} \subseteq T(V) \subseteq T(X)$. Since $T(X)$ is a vector subspace of Y, the latter inclusion implies that $T(X) = Y$ must hold.

Problem 28.2. *Let X be a Banach space, $T\colon X \to X$ a bounded operator, and I the identity operator on X. If $\|T\| < 1$, then show that $I - T$ is invertible.*

Solution. If $A, B \in L(X, X)$, then the inequalities

$$\|ABx\| \le \|A\| \cdot \|Bx\| \le \|A\| \cdot \|B\| \cdot \|x\|$$

easily imply that $\|AB\| \le \|A\| \cdot \|B\|$. In particular, if a sequence $\{A_n\}$ of operators of $L(X, X)$ satisfies $\lim A_n = A$ in $L(X, X)$ and $B \in L(X, X)$, then the inequality

$$\|BA_n - BA\| = \|B(A_n - A)\| \le \|B\| \cdot \|A_n - A\|$$

shows that $\lim BA_n = BA$. Similarly, $\lim A_n B = AB$.

Now, assume $T \in L(X, X)$ satisfies $\|T\| < 1$. In view of the inequality $\|T^n\| \le \|T\|^n$, it follows that

$$\sum_{n=0}^{\infty} \|T^n\| \le \sum_{n=0}^{\infty} \|T\|^n = \tfrac{1}{1-\|T\|} < \infty.$$

Thus, $\sum_{n=0}^{\infty} T^n$ is an absolutely summable series. Since $L(X, X)$ is a Banach space, $S = \sum_{n=0}^{\infty} T^n$ converges in $L(X, X)$; see Problem 27.7. Moreover,

$$(I - T)S = \lim_{n\to\infty} (I - T)\Big(\sum_{i=0}^{n} T^i\Big) = \lim_{n\to\infty} \big(I - T^{n+1}\big) = I,$$

and similarly $S(I - T) = I$. Therefore, $S = (I - T)^{-1}$.

Problem 28.3. *On $C[0, 1]$ consider the two norms*

$$\|f\|_\infty = \sup\{|f(x)|: x \in [0, 1]\} \quad \textit{and} \quad \|f\|_1 = \int_0^1 |f(x)|\, dx.$$

Then show that the identity operator $I: (C[0, 1], \|\cdot\|_\infty) \to (C[0, 1], \|\cdot\|_1)$ is continuous, onto, but not open. Why doesn't this contradict the Open Mapping Theorem?

Solution. Clearly, I is onto, and in view of the inequality $\|f\|_1 \le \|f\|_\infty$, we see that I is also continuous.

For the rest of the proof, we need to show that $\big(C[0, 1], \|\cdot\|_1\big)$ is not a Banach space. To establish this, consider the sequence $\{f_n\}$ of continuous functions whose graphs are shown in Figure 5.1.

The inequality $\|f_{n+p} - f_n\|_1 \le \frac{1}{n}$ shows that $\{f_n\}$ is a Cauchy sequence for the norm $\|\cdot\|_1$. Assume by way of contradiction that $\lim \|f_n - f\|_1 = 0$ holds for some $f \in C[0, 1]$.

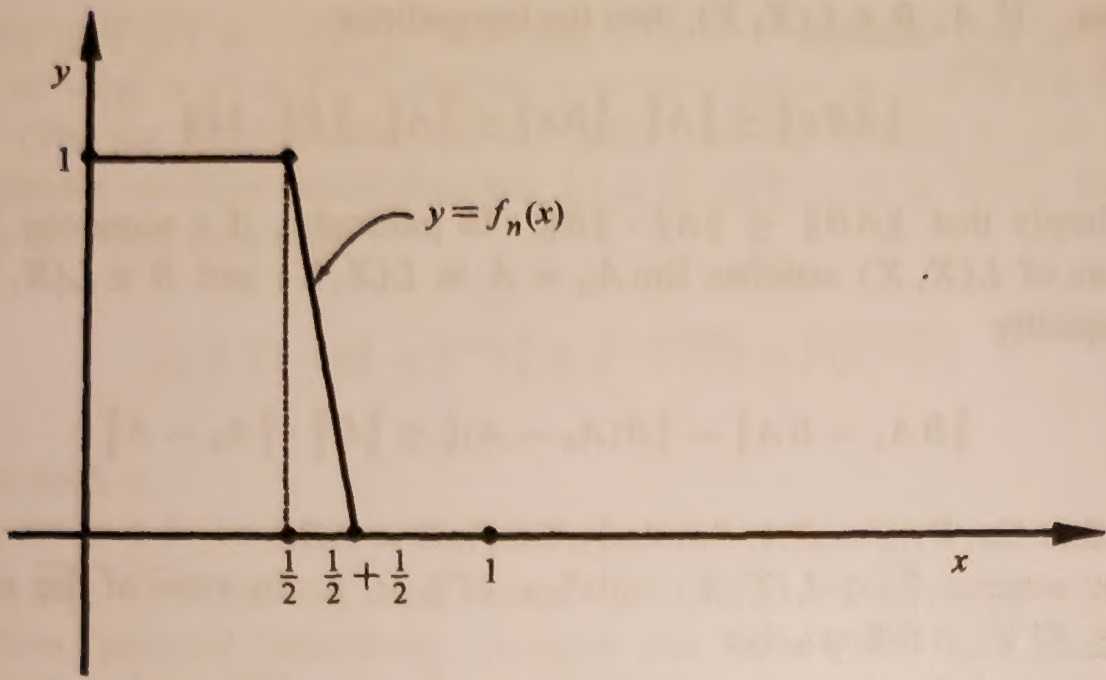

FIGURE 5.1.

Let $a \in (0, \frac{1}{2})$. If $f(a) \neq 1$, then there exist some $\varepsilon > 0$ and some $0 < \delta < \min\{a, \frac{1}{2} - a\}$ such that $|f(x) - 1| \geq \varepsilon$ holds whenever $|x - a| < \delta$. Now, note that $2\delta\varepsilon \leq \int_0^1 |f_n(x) - f(x)|\,dx = \|f_n - f\|_1$ for all sufficiently large n, contrary to $\lim \|f_n - f\|_1 = 0$. Thus, $f(a) = 1$ holds for all $a \in (0, \frac{1}{2})$. Similarly, $f(a) = 0$ for all $a \in (\frac{1}{2}, 1)$. Now, it is readily seen that f cannot be a continuous function, contrary to $f \in C[0, 1]$. Thus, $\{f_n\}$ does not converge in $C[0, 1]$ with respect to the $\|\cdot\|_1$ norm.

Finally, $I\colon (C[0, 1], \|\cdot\|_\infty) \longrightarrow (C[0, 1], \|\cdot\|_1)$ cannot be an open mapping. Since otherwise, $\|\cdot\|_1$ and $\|\cdot\|_\infty$ would be equivalent norms, and therefore $(C[0, 1], \|\cdot\|_1)$ would be a Banach space.

Problem 28.4. *Let X be the vector space of all real-valued functions on $[0, 1]$ that have continuous derivatives with the sup norm. Also, let $Y = C[0, 1]$ with the sup norm. Define $D\colon X \to Y$ by $D(f) = f'$.*

a. *Show that D is an unbounded linear operator.*
b. *Show that D has a closed graph.*
c. *Why doesn't the conclusion in (b) contradict the Closed Graph Theorem?*

Solution. (a) The standard properties of differentiation guarantee that D is a linear operator. Now, for each n let $f_n(x) = x^n$. Then, $f_n \in X$ and $\|f_n\|_\infty = \sup\{|f_n(x)|\colon 0 \leq x \leq 1\} = 1$ for each n. Now, notice that $D(f_n)(x) = nx^{n-1}$ holds for each n, and from this it follows that

$$\|D\| \geq \|D(f_n)\|_\infty = \sup\{nx^{n-1}\colon 0 \leq x \leq 1\} = n,$$

Therefore, $\|D\| = \infty$, and so D is an unbounded operator.

(b) To see that D has a closed graph, assume $f_n \to 0$ in X and $Df_n = f_n' \to g$ in Y. That is, $\{f_n\}$ converges uniformly to zero, and $\{f_n'\}$ converges uniformly to g. We have to show that $g = 0$.

From $\int_0^x f_n'(t)\,dt = f_n(x) - f_n(0)$ (and Problem 9.16), it follows that

$$\int_0^x g(t)\,dt = \lim_{n\to\infty} \int_0^x f_n'(t)\,dt = \lim_{n\to\infty} \big[f_n(x) - f_n(0)\big] = 0$$

holds for all $x \in [0, 1]$. Differentiating, we get $g(x) = 0$ for each $x \in [0, 1]$, as required. (See also Problem 9.29.)

(c) The conclusion in (b) does not contradict the Closed Graph Theorem since X is not a Banach space. For instance, we know (from Corollary 11.6) that every function $f \in C[0, 1]$ is the uniform limit of a sequence of polynomials. So, if $f \in C[0, 1]$ is a nondifferentiable continuous function and $\{p_n\}$ is a sequence of polynomials that converges uniformly to f, then $\{p_n\}$ is a Cauchy sequence of X which cannot converge in X.

Problem 28.5. *Consider the mapping $T: C[0, 1] \to C[0, 1]$ defined by $Tf(x) = x^2 f(x)$ for all $f \in C[0, 1]$ and each $x \in [0, 1]$.*

a. *Show that T is a bounded linear operator.*
b. *If $I: C[0, 1] \to C[0, 1]$ denotes the identity operator (i.e., $I(f) = f$ for each $f \in C[0, 1]$), then show that $\|I + T\| = 1 + \|T\|$.*

Solution. (a) From the identities

$$T(f+g)(x) = x^2(f+g)(x) = x^2 f(x) + x^2 g(x) = \big(Tf + Tg\big)(x)$$

and

$$T(\alpha f)(x) = \alpha x^2 f(x) = \big(\alpha T f\big)(x),$$

we easily infer that T is a linear operator. For the norm of T, note that for each $f \in C[0, 1]$ we have

$$\|Tf\|_\infty = \sup_{x\in[0,1]} |Tf(x)| = \sup_{x\in[0,1]} x^2|f(x)| \le \sup_{x\in[0,1]} |f(x)| = \|f\|_\infty,$$

and so $\|T\| \le 1$. On the other hand, for the constant function $\mathbf{1}$, we have

$$\|T\| \ge \|T\mathbf{1}\|_\infty = \sup_{x\in[0,1]} x^2 = 1.$$

Thus, $\|T\| = 1$, and so T is a bounded operator.

(b) Clearly, $\|I+T\| \le \|I\| + \|T\| = 1 + 1 = 2$. Moreover, we have

$$\|I+T\| \ge \|(I+T)\mathbf{1}\|_\infty = \sup_{x\in[0,1]} (1+x^2) = 2,$$

and so $\|I+T\| = 1 + \|T\| = 2$ holds true.

Problem 28.6. *Let X be a vector space which is complete in each of the two norms $\|\cdot\|_1$ and $\|\cdot\|_2$. If there exists a real number $M > 0$ such that $\|x\|_1 \le M\|x\|_2$ holds for all $x \in X$, then show that the two norms must be equivalent.*

Solution. The identity operator $I\colon (X, \|\cdot\|_2) \longrightarrow (X, \|\cdot\|_1)$ is one-to-one, continuous, and onto. By the Open Mapping Theorem it is a homeomorphism, and the conclusion follows.

Problem 28.7. *Let X, Y, and Z be three Banach spaces. Assume that $T\colon X \to Y$ is a linear operator and $S\colon Y \to Z$ is a bounded, one-to-one linear operator. Show that T is a bounded operator if and only if the composite linear operator $S \circ T$ (from X into Z) is bounded.*

Solution. Assume that $S \circ T$ is a bounded operator. Let $x_n \to 0$ in X and $T(x_n) \to y$ in Y. Using that $S \circ T$ and S are both continuous, we get

$$S(y) = \lim_{n\to\infty} S\big(T(x_n)\big) = \lim_{n\to\infty} S \circ T(x_n) = 0.$$

Since S is one-to-one, we infer that $y = 0$, and hence—by the Closed Graph Theorem—the operator T is continuous.

Problem 28.8. *An operator $P\colon V \to V$ on a vector space is said to be a* **projection** *if $P^2 = P$ holds. Also, a closed vector subspace Y of a Banach space is said to be* **complemented** *if there exists another closed subspace Z of X such that $Y \oplus Z = X$.*

Show that a closed subspace of a Banach space is complemented if and only if it is the range of a continuous projection.

Solution. Let Y be a closed subspace of a Banach space X. Assume first that there exists a continuous projection $P\colon X \to X$ whose range is Y, i.e., $P(X) = Y$. From $P^2 = P$, it follows that $Y = \{y \in X\colon\ y = Py\}$.

If $I\colon X \to X$ denotes the identity operator, let $Z = (I - P)(X)$, the range of the continuous operator $I - P$. Clearly, Z is a vector subspace of X and in view of $x = Px + (I-P)(x)$, we see that $Y + Z = X$. Now, if $u \in Y \cap Z$, then

$u = z - Pz$ for some $z \in Z$, and so $u = P(u) = P(z - Pz) = P(z) - P^2(z) = 0$. This means that $Y \oplus Z = X$. Finally, to see that Z is also closed, assume that a sequence $\{z_n\}$ of X satisfies $(I - P)(z_n) \to z$. Then, the continuity of P implies $0 = P(I - P)(z_n) \to Pz$, and so $Pz = 0$. Hence, $z = (I - P)(z) \in Z$, proving that Z is also closed. Thus, Y is a complemented closed subspace.

For the converse, assume that Z is a closed subspace such that $Y \oplus Z = X$. So, for each $x \in X$ there exist $y \in Y$ and $z \in Z$ (both uniquely determined) such that $x = y + z$. Define an operator $P: X \to X$ via the formula $P(x) = y$, where y satisfies $x - y \in Z$. We claim that P is a continuous projection whose range is Y.

Notice first that P is a linear operator. Also, $P^2(x) = P(y) = y = P(x)$ holds for each $x \in X$, so that P is a projection. Clearly, the range of P is Y. To finish the proof, we must show that P is also continuous. For this, it suffices to show (in view of the Closed Graph Theorem) that P has a closed graph.

To this end, assume that a sequence $\{x_n\}$ of X satisfies $x_n \to x$ and $P(x_n) \to y$ in X. For each n let $x_n = y_n + z_n$, where $y_n \in Y$ and $z_n \in Z$. Clearly, $y_n = P(x_n)$ for each n. Since Y is a closed subspace, it follows that $y \in Y$. Now, from $z_n = x_n - y_n \to x - y$ and the closedness of Z, we infer that $z = x - y \in Z$. Thus, $x = y + z$, and so $y = P(x)$. This shows that P has a closed graph, and we are done.

29. LINEAR FUNCTIONALS

Problem 29.1. *Let $f: X \to \mathbf{R}$ be a linear functional defined on a vector space X. The* **kernel** *of f is the vector subspace*

$$\operatorname{Ker} f = f^{-1}(\{0\}) = \{x \in X: f(x) = 0\}.$$

If X is a normed space and $f: X \to \mathbf{R}$ is nonzero linear functional, establish the following:

a. *f is continuous if and only if its kernel is a closed subspace of X.*
b. *f is discontinuous if and only if its kernel is dense in X.*

Solution. (a) Clearly, if f is continuous, then its kernel $f^{-1}(\{0\})$ is a closed set. For the converse, assume that $f \neq 0$ and that $f^{-1}(\{0\})$ is a closed set. Pick some $e \in X$ with $f(e) = 1$.

Suppose by way of contradiction that $\|f\| = \infty$. Then, there exists a sequence $\{x_n\}$ of X with $\|x_n\| = 1$ and $|f(x_n)| \geq n$ for each n. Note that the sequence $\{y_n\}$, defined by $y_n = e - \frac{x_n}{f(x_n)}$, satisfies $y_n \in f^{-1}(\{0\})$ for each n and $y_n \longrightarrow e$. Since the set $f^{-1}(\{0\})$ is closed, it follows that $e \in f^{-1}(\{0\})$, and so $f(e) = 0$, which is a contradiction. Thus, f is a continuous linear functional.

(b) If $\operatorname{Ker} f$ is dense in X, then $\operatorname{Ker} f$ is not closed and hence, by part (a), f is not continuous. For the converse, assume that f is a discontinuous linear functional, i.e., $\|f\| = \infty$. This implies (as in the previous part) that there exists a sequence $\{x_n\}$ of X satisfying $\|x_n\| = 1$ and $|f(x_n)| \geq n$ for each n. Now, if $x \in X$ and $y_n = x - \frac{f(x)}{f(x_n)} x_n$, then $\{y_n\}$ is a sequence in $\operatorname{Ker} f$ and satisfies $y_n \to x$. This shows that $\operatorname{Ker} f$ is dense in X.

Problem 29.2. *Show that a linear functional f on a normed space X is discontinuous if and only if for each $a \in X$ and each $r > 0$, we have*

$$f(B(a,r)) = \{f(x): \|a - x\| < r\} = \mathbb{R}.$$

Solution. Let f be a linear functional on a normed space X and let $B = B(0,1) = \{x \in X: \|x\| < 1\}$. Assume that f is discontinuous. Fix $a \in X$ and $r > 0$. From the relation $B(a,r) = a + rB(0,1) = a + rB$ and the linearity of f, it follows that $f\big(B(a,r)\big) = \mathbb{R}$ holds if and only if $f(B) = \Re$.

We claim first that $f(B)$ is unbounded from above in $\mathbb{R}$. To see this, assume by way of contradiction that there exists some $M > 0$ such that $f(x) \leq M$ holds for each $x \in B$. Note that if $x \in X$ satisfies $\|x\| \leq 1$, then $\pm\frac{1}{2}x \in B$, and so from

$$\pm\tfrac{1}{2} f(x) = f\big(\pm\tfrac{1}{2}x\big) \leq \tfrac{M}{2},$$

we see that $|f(x)| \leq M$ holds for all $x \in X$ with $\|x\| \leq 1$. That is, $\|f\| = \sup\big\{|f(x)|: \|x\| \leq 1\big\} \leq M < \infty$, and so f is a continuous linear functional, a contradiction. Thus, $f(B)$ is unbounded from above in $\mathbb{R}$. Now, let $\alpha > 0$ be an arbitrary positive real number. By the above, there exists some $x \in B$ satisfying $f(x) > \alpha$. Now, note that the element $y = \frac{\alpha}{f(x)} x \in B$ satisfies $f(y) = \alpha$ (and, of course, $-y \in B$ satisfies $f(-y) = -\alpha$). Consequently, $f(B) = \mathbb{R}$.

For the converse, assume that $f\big(B(a,r)\big) = \mathbb{R}$ holds for each $a \in X$ and each $r > 0$. In particular, from

$$\infty = \sup\big\{|f(x)|: \|x\| < \tfrac{1}{2}\big\} \leq \sup\big\{|f(x)|: \|x\| \leq 1\big\} = \|f\|,$$

we see that $\|f\| = \infty$. Thus, f is unbounded and so (by Theorem 28.6) f is a discontinuous linear functional.

Problem 29.3. *Let $f, f_1, f_2, \ldots, f_n$ be linear functionals defined on a common vector space X. Show that there exist constants $\lambda_1, \ldots, \lambda_n$ satisfying $f = \sum_{i=1}^n \lambda_i f_i$ (i.e., f lies in the linear span of $f_1, \ldots, f_n$) if and only if $\bigcap_{i=1}^n \operatorname{Ker} f_i \subseteq \operatorname{Ker} f$.*

Solution. If $f = \sum_{i=1}^{n} \lambda_i f_i$ holds, then clearly $\bigcap_{i=1}^{n} \operatorname{Ker} f_i \subseteq \operatorname{Ker} f$. For the converse, assume $\bigcap_{i=1}^{n} \operatorname{Ker} f_i \subseteq \operatorname{Ker} f$. Let

$$V = \{y \in \mathbf{R}^n \colon \exists\ x \in X \text{ such that } y = (f_1(x), f_2(x), \ldots, f_n(x))\}.$$

It is easy to verify that V is a vector subspace of $\mathbf{R}^n$. Now, define the linear functional $g\colon V \to \mathbf{R}$ via the formula

$$g(f_1(x), f_2(x), \ldots, f_n(x)) = f(x).$$

Notice that g is well defined. To see this, assume

$$(f_1(x), f_2(x), \ldots, f_n(x)) = (f_1(y), f_2(y), \ldots, f_n(y)).$$

Then, $f_i(x-y) = 0$ for each i, and so $x - y \in \bigcap_{i=1}^{n} \operatorname{Ker} f_i$. From our hypothesis, it follows that $x - y \in \operatorname{Ker} f$, which means that $f(x) = f(y)$. Now, it is a routine matter to verify that g is linear.

Denote by g again a linear extension of g to all of $\mathbf{R}^n$. This implies that there exist scalars $\lambda_1, \ldots, \lambda_n$ such that $g(z_1, \ldots, z_n) = \sum_{i=1}^{n} \lambda_i z_i$ holds for all $(z_1, \ldots, z_n) \in \mathbf{R}^n$. In particular, we have

$$f(x) = g(f_1(x), f_2(x), \ldots, f_n(x)) = \sum_{i=1}^{n} \lambda_i f_i(x)$$

for all $x \in X$, as desired.

Problem 29.4. *Prove the converse of Theorem 28.7. That is, show that if X and Y are (nontrivial) normed spaces and $L(X, Y)$ is a Banach space, then Y is a Banach space.*

Solution. Let $\{y_n\}$ be a Cauchy sequence of Y. Pick some f in X^* with $f \neq 0$, and then consider the sequence of operators $\{T_n\}$ of $L(X, Y)$ defined by $T_n(x) = f(x)y_n$. The inequality

$$\|T_n(x) - T_m(x)\| = \|f(x)(y_n - y_m)\| \le \|f\| \cdot \|y_n - y_m\| \cdot \|x\|,$$

shows that $\|T_n - T_m\| \le \|f\| \cdot \|y_n - y_m\|$, and so $\{T_n\}$ is a Cauchy sequence of $L(X, Y)$. By the completeness of $L(X, Y)$, there exists some $T \in L(X, Y)$ with $\lim T_n = T$. Now, pick some $e \in X$ with $f(e) = 1$, and note that

$$\lim_{n\to\infty} T_n(e) = \lim_{n\to\infty} y_n = T(e).$$

Problem 29.5. *The Banach space $B(\mathbb{N})$ is denoted by ℓ_∞. That is, ℓ_∞ is the Banach space consisting of all bounded sequences with the sup norm. Consider the collections of vectors*

$$c_0 = \{x = (x_1, x_2, x_3, \ldots) \in \ell_\infty\colon\ x_n \to 0\}, \text{ and}$$
$$c = \{x = (x_1, x_2, x_3, \ldots) \in \ell_\infty\colon\ \lim x_n \text{ exists in } \mathbb{R}\}.$$

Show that c_0 and c are both closed vector subspaces of ℓ_∞.

Solution. It should be obvious that c_0 and c are vector subspaces of ℓ_∞ (and that c_0 is a vector subspace of c). What needs verification is their closedness. To see that c_0 is closed, assume that a sequence $\{x^n\}$ of c_0, where $x^n = (x_1^n, x_2^n, \ldots)$, satisfies $\|x^n - x\|_\infty \to 0$. If $x = (x_1, x_2, \ldots)$, we must show that $\lim x_n = 0$.

To this end, let $\epsilon > 0$. Fix some k such that $\|x^n - x\| < \epsilon$ for all $n \geq k$; clearly $|x_i^n - x_i| < \epsilon$ also holds for all $n \geq k$ and all i. Since $\lim_{i\to\infty} x_i^k = 0$, there exists some $m \geq k$ such that $|x_i^k| < \epsilon$ holds for all $i \geq m$. Now, notice that if $i \geq m$, then

$$|x_i| \leq |x_i - x_i^k| + |x_i^k| < \epsilon + \epsilon = 2\epsilon.$$

This shows that $\lim x_n = 0$, as desired.

Next, we shall establish that c is closed. For simplicity, for a sequence $x = (x_1, x_2, \ldots) \in c$ we shall write $x_\infty = \lim x_n$. Now, assume that a sequence $\{x^n\}$ in c satisfies $x^n \to x = (x_1, x_2, \ldots) \in \ell_\infty$. We must show that $\lim x_n$ exists in $\mathbb{R}$.

Start by fixing some $\epsilon > 0$. Then, there exists some k such that

$$\|x^n - x\|_\infty < \epsilon \quad \text{holds for all} \quad n \geq k. \tag{$\star$}$$

This implies $\|x^n - x^m\|_\infty < 2\epsilon$ for all $n, m \geq k$, and so $|x_i^n - x_i^m| < 2\epsilon$ for all $n, m \geq k$ and each i. Consequently, $|x_\infty^n - x_\infty^m| \leq 2\epsilon$ for all $n, m \geq k$. This shows that $\{x_\infty^n\}$ is a Cauchy sequence of real numbers. Let $x_\infty = \lim x_\infty^n$ and note that $|x_\infty^n - x_\infty| \leq 2\epsilon$ holds for all $n \geq k$.

We claim that $x_n \to x_\infty$. To see this, let again $\epsilon > 0$ and choose k so that ($\star$) is true. Next, fix some $r \geq k$ such that $|x_n^k - x_\infty^k| < \epsilon$ holds for all $n \geq r$. Now, note that if $n \geq r$, then

$$|x_n - x_\infty| \leq |x_n - x_n^k| + |x_n^k - x_\infty^k| + |x_\infty^k - x_\infty| < \epsilon + \epsilon + 2\epsilon = 4\epsilon.$$

This shows that $x_n \to x_\infty$, and so $x \in c$. Therefore, c is a closed subspace of ℓ_∞.

Problem 29.6. *Let c denote the vector subspace of ℓ_∞ consisting of all convergent sequences (see Problem 29.5). Define the* **limit functional** $L\colon c \to \mathbb{R}$

by

$$L(x) = L(x_1, x_2, \ldots) = \lim_{n\to\infty} x_n,$$

and $p\colon \ell_\infty \to \mathbf{R}$ *by* $p(x) = p(x_1, x_2, \ldots) = \limsup x_n$.

a. *Show that* L *is a continuous linear functional, where* c *is assumed equipped with the sup norm.*
b. *Show that* p *is sublinear and that* $L(x) = p(x)$ *holds for each* $x \in c$.
c. *By the Hahn–Banach Theorem 29.2 there exists a linear extension of* L *to all of* ℓ_∞ *(which we shall denote by* L *again) satisfying* $L(x) \le p(x)$ *for all* $x \in \ell_\infty$. *Establish the following properties of the extension* L*:*
   i. *For each* $x \in \ell_\infty$, *we have*

$$\liminf_{n\to\infty} x_n \le L(x) \le \limsup_{n\to\infty} x_n.$$

   ii. L *is a positive linear functional, i.e.,* $x \ge 0$ *implies* $L(x) \ge 0$.
   iii. L *is a continuous linear functional (and in fact* $\|L\| = 1$*).*

Solution. (a) Clearly, L is a linear functional. Moreover, if $x = (x_1, x_2, \ldots) \in c$, then $|x_n| \le \|x\|_\infty = \sup_m |x_m|$ for each n, and so $|L(x)| = \lim |x_n| \le \|x\|_\infty$. This shows that L is a continuous linear functional. (Since $L(1, 1, 1, \ldots) = 1$, it is easy to see that $\|L\| = 1$.)
(b) The sublinearity of p follows immediately from Problem 4.7. The equality $p(x) = L(x) = \lim x_n$ for each $x \in c$ should be also obvious.
(c) We shall establish the stated properties.

(i) If $x \in \ell_\infty$, then notice that

$$-L(x) = L(-x) \le \limsup_{n\to\infty}(-x_n) = -\liminf_{n\to\infty} x_n,$$

and so $\liminf x_n \le L(x) \le \limsup x_n$ holds true.

(ii) If $x = (x_1, x_2, \ldots) \ge 0$ (i.e., if $x_n \ge 0$ for each n), then it follows from Problem 4.8 and the preceding conclusion that $L(x) \ge \liminf x_n \ge 0$. That is, L is a positive linear functional.

(iii) If $\|x\|_\infty \le 1$ (i.e., if $|x_n| \le 1$ for each n), then it follows from part (i) and Problem 4.8 that

$$-1 \le \liminf_{n\to\infty} x_n \le L(x) \le \limsup_{n\to\infty} x_n \le 1,$$

and so $|L(x)| \le 1$. This implies $\|L\| = \sup\{|L(x)|\colon \|x\|_\infty \le 1\} \le 1$. Since $L(1, 1, 1, \ldots) = 1$, we easily infer that $\|L\| = 1$.

Problem 29.7. *Generalize Problem 29.6 as follows. Show that there exists a linear functional* $\mathcal{L}im\colon \ell_\infty \to \mathbf{R}$ *(called a* **Banach–Mazur limit***) with the following properties.*

a. $\mathcal{L}im$ *is a positive linear functional of norm one.*

b. *For each* $x = (x_1, x_2, \ldots) \in \ell_\infty$*, we have*

$$\liminf_{n\to\infty} \frac{x_1 + x_2 + \cdots + x_n}{n} \le \mathcal{L}im(x) \le \limsup_{n\to\infty} \frac{x_1 + x_2 + \cdots + x_n}{n}.$$

In particular, $\mathcal{L}im$ *is an extension of the limit functional* L.

c. *For each* $x = (x_1, x_2, \ldots) \in \ell_\infty$*, we have*

$$\mathcal{L}im(x_1, x_2, x_3, \ldots) = \mathcal{L}im(x_2, x_3, x_4, \ldots).$$

Solution. For each $x = (x_1, x_2, \ldots) \in \ell_\infty$, let

$$A(x) = \left(x_1, \frac{x_1 + x_2}{2}, \ldots, \frac{x_1 + x_2 + \cdots + x_n}{n}, \ldots\right)$$

be the sequence of averages of x. If we define $p\colon \ell_\infty \to \mathbf{R}$ via the formula

$$p(x) = \limsup A(x) = \limsup_{n\to\infty} \frac{x_1 + x_2 + \cdots + x_n}{n},$$

then a glance at Problem 4.7 guarantees that p is a sublinear functional. Moreover, it is easy to see that if $x \in c$, then

$$L(x) = \lim_{n\to\infty} x_n = p(x).$$

Now, by the Hahn–Banach Theorem 29.2, L has an extension $\mathcal{L}im\colon \ell_\infty \to \mathbf{R}$ satisfying $\mathcal{L}im(x) \le p(x)$ for each $x \in \ell_\infty$. Properties (a) and (b) can be established exactly as in the solution of Problem 29.6.

To verify (c), let $x = (x_1, x_2, \ldots) \in \ell_\infty$ and put $y = (x_2, x_3, \ldots)$. Then, an easy computation shows that

$$A(x - y) = \left(x_1 - x_2, \frac{x_1 - x_3}{2}, \frac{x_1 - x_4}{3}, \ldots, \frac{x_1 - x_{n+1}}{n}, \ldots\right).$$

Since $x = (x_1, x_2, \ldots)$ is a bounded sequence, the latter implies

$$p(x - y) = \limsup \frac{x_1 - x_{n+1}}{n} = 0.$$

Hence, $\mathcal{L}im(x-y) \le p(x-y) = 0$. Similarly, $L(y-x) \le 0$, and so $\mathcal{L}im(x) - \mathcal{L}im(y) = \mathcal{L}im(x-y) = 0$. Thus, $\mathcal{L}im(x) = \mathcal{L}im(y)$, as desired.

Problem 29.8. *Let X be a normed vector space. Show that if X^* is separable (in the sense that it contains a countable dense subset), then X is also separable.*

Solution. Let $\{f_1, f_2, \ldots\}$ be a countable dense subset of X^*. For each n choose some $x_n \in X$ with $\|x_n\| = 1$ and $|f_n(x_n)| \ge \frac{1}{2}\|f_n\|$, and let Y be the closed subspace generated by $\{x_1, x_2, \ldots\}$. We claim that $Y = X$.

To see this, assume by way of contradiction that $Y \ne X$. Fix some $a \notin Y$ with $\|a\| = 1$. By Theorem 29.5, there exists some $f \in X^*$ with $f(y) = 0$ for all $y \in Y$ and $f(a) \ne 0$. Given $\varepsilon > 0$ choose some n with $\|f - f_n\| < \varepsilon$, and note that

$$\big|f_n(a)\big| \le \big\|f_n\big\| \le 2\big|f_n(x_n)\big| = 2\big|(f_n - f)(x_n)\big| \le 2\big\|f_n - f\big\| < 2\varepsilon.$$

Thus, $|f(a)| \le |f(a) - f_n(a)| + |f_n(a)| < 3\varepsilon$ holds for all $\varepsilon > 0$, and so $f(a) = 0$, a contradiction. Therefore, $Y = X$ holds.

Now, note that the collection of all finite linear combinations of the countable set $\{x_1, x_2, \ldots\}$ with rational coefficients is a countable dense subset of X.

Problem 29.9. *Show that a Banach space X is reflexive if and only if X^* is reflexive.*

Solution. Assume that X^* is reflexive. If $X \ne X^{**}$, then by Theorem 29.5 there exists some nonzero $F \in X^{***}$ with $F(x) = 0$ for each $x \in X$. Since X^* is reflexive, there exists a nonzero $x^* \in X^*$ so that $F(f) = f(x^*)$ holds for all $f \in X^{**}$. In particular,

$$x^*(x) = \hat{x}(x^*) = F(x) = 0$$

holds for all $x \in X$, and so $x^* = 0$, a contradiction. Therefore, X must be a reflexive Banach space.

Problem 29.10. *This problem describes the adjoint of a bounded operator. If $T: X \to Y$ is a bounded operator between two normed spaces, then the* **adjoint** *of T is the operator $T^*: Y^* \to X^*$ defined by $(T^*f)(x) = f(Tx)$ for all $f \in Y^*$ and all $x \in X$. (Writing $h(x) = \langle x, h\rangle$, the definition of the adjoint operator is written in "duality" notation as*

$$\langle Tx, f\rangle = \langle x, T^*f\rangle$$

for all $f \in Y^*$ and all $x \in X$.)

a. *Show that $T^*: Y^* \to X^*$ is a well-defined bounded linear operator whose norm coincides with that of T, i.e., $\|T^*\| = \|T\|$.*

b. *Fix some $g \in X^*$ and some $u \in Y$ and define $S: X \to Y$ by $S(x) = g(x)u$. Show that S is a bounded linear operator satisfying $\|S\| = \|g\| \cdot \|u\|$. (Any such operator S is called a* **rank-one** *operator.)*

c. *Describe the adjoint of the operator S defined in part (b).*

d. *Let $A = [a_{ij}]$ be an $m \times n$ matrix with real entries. As usual, we consider the adjoint A^* as a (bounded) linear operator from $\mathbf{R}^n$ to $\mathbf{R}^m$. Describe A^*.*

Solution. As usual, we shall denote from simplicity $T(x)$ by Tx.

(a) Fix $f \in Y^*$. Then, for $x, y \in X$ and $\alpha, \beta \in \mathbf{R}$, we have

$$\begin{aligned}(T^*f)(\alpha x + \beta y) &= f(T(\alpha x + \beta y)) = f(\alpha Tx + \beta Ty)\\ &= \alpha f(Tx) + \beta f(Ty) = \alpha(T^*f)(x) + \beta(T^*f)(y),\end{aligned}$$

so that T^*f is a linear functional on X. To see that T^*f is also continuous, notice that

$$\left|(T^*f)(x)\right| = \left|f(Tx)\right| \le \|f\| \cdot \|T(x)\| \le \|f\| \cdot \|T\| \cdot \|x\|$$

holds for all $x \in X$. This shows that T^*f is a bounded (and hence, continuous) linear functional and that $\|T^*f\| \le \|T\| \cdot \|f\|$ holds true for each $f \in Y^*$.

The last inequality also shows that $T^*: Y^* \to X^*$ is a bounded operator and that $\|T^*\| \le \|T\|$. For the reverse inequality, let $x \in X$ satisfy $\|x\| \le 1$. By Theorem 29.4 there exists some $h \in Y^*$ satisfying $\|h\| = 1$ and $h(Tx) = \|Tx\|$. So,

$$\|T^*\| \ge \|T^*h\| \ge \|T^*h(x)\| = \left\|h(Tx)\right\| = \|Tx\|,$$

for each $x \in X$ with $\|x\| \le 1$. This implies $\|T\| = \sup\{\|Tx\|: \|x\| \le 1\} \le \|T^*\|$. Hence, $\|T^*\| = \|T\|$.

(b) It is a routine matter to verify that S is linear. From

$$\begin{aligned}\|S(x)\| &= \|g(x)u\| = \left|g(x)\right| \cdot \|u\|\\ &\le \|g\| \cdot \|x\| \cdot \|u\| = \big(\|g\| \cdot \|u\|\big)\|x\|,\end{aligned}$$

we see that S is a bounded operator and that $\|S\| \le \|g\| \cdot \|u\|$. Now, if $x \in X$

satisfies $\|x\| \le 1$, then we have

$$\|S\| \ge \|S(x)\| = \|g(x)u\| \ge |g(x)| \cdot \|u\|,$$

and so $\|S\| \ge \sup\{|g(x)| \cdot \|u\|: x \in X \text{ and } \|x\| \le 1\} = \|g\| \cdot \|u\|$. The preceding show that $\|S\| = \|g\| \cdot \|u\|$.

(c) Note that for each $f \in Y^*$ and each $x \in X$ we have

$$(S^* f)(x) = f(Sx) = f(g(x)u) = f(u)g(x) = \big[f(u)g\big](x).$$

So, $S^* f = f(u)g$ holds for all $f \in Y^*$.

(d) Let $A = [a_{ij}]$ be an $m \times n$ real matrix. Note that the norm dual of $\mathbf{R}^n$ is again $\mathbf{R}^n$, where every $y = (y_1, \ldots, y_n) \in \mathbf{R}^n$ defines a linear functional on $\mathbf{R}^n$ via the formula

$$y(x) = \langle x, y\rangle = x \cdot y = \sum_{i=1}^{n} x_i y_i .$$

This easily implies that the adjoint A^* of A is an $n \times m$ matrix $B = [b_{ij}]$ that satisfies the duality identity $\langle Ax, y\rangle = \langle x, A^* y\rangle$, or $Ax \cdot y = x \cdot By$. That is, the elements of B satisfy the equation

$$\sum_{j=1}^{m} \sum_{i=1}^{n} a_{ij} x_i y_j = \sum_{i=1}^{n} \sum_{j=1}^{m} b_{ji} y_j x_i$$

for all $x \in \mathbf{R}^n$ and all $y \in \mathbf{R}^m$. This easily implies $b_{ij} = a_{ji}$ for all i and j. Therefore, A^* is the transpose of A, i.e. $A^* = A^t$.

30. BANACH LATTICES

Problem 30.1. *Let X be a vector lattice, and let $f: X^+ \to [0, \infty)$ be an additive function (that is, $f(x + y) = f(x) + f(y)$ holds for all $x, y \in X^+$). Then show that there exists a unique linear functional g on X such that $g(x) = f(x)$ holds for all $x \in X^+$.*

Solution. Note first that if $x \ge y \ge 0$ holds, then

$$f(x) = f\big(y + (x - y)\big) = f(y) + f(x - y) \ge f(y).$$

Also, the arguments of the proof of Lemma 18.7 show that $f(rx) = rf(x)$ holds for all $x \in X^+$ and all rational numbers $r \ge 0$.

Now, let $\alpha > 0$ and $x \geq 0$. Pick two sequences $\{r_n\}$ and $\{t_n\}$ of rational numbers with $0 \leq r_n \uparrow \alpha$ and $t_n \downarrow \alpha$. Then, the inequality $r_n x \leq \alpha x \leq t_n x$ implies

$$r_n f(x) = f(r_n x) \leq f(\alpha x) \leq f(t_n x) = t_n f(x),$$

from which it follows that $\alpha f(x) = f(\alpha x)$ holds.

Now, define $g: X \longrightarrow \mathbb{R}$ by

$$g(x) = f(x^+) - f(x^-).$$

Note that if $x = y - z$ holds with $y,\ z \in X^+$, then the relation $x^+ + z = y + x^-$, coupled with the additivity of f on X^+, shows that $f(x^+)+f(z) = f(y)+f(x^-)$. That is,

$$g(x) = f(x^+) - f(x^-) = f(y) - f(z).$$

In particular, for $x,\ y \in X$ we have

$$\begin{aligned} g(x+y) &= g\big(x^+ + y^+ - (x^- + y^-)\big) = f(x^+ + y^+) - f(x^- + y^-) \\ &= f(x^+) + f(y^+) - f(x^-) - f(y^-) \\ &= \big[f(x^+) - f(x^-)\big] + \big[f(y^+) - f(y^-)\big] \\ &= g(x) + g(y). \end{aligned}$$

Moreover, for $\alpha > 0$ we have

$$g(\alpha x) = f(\alpha x^+) - f(\alpha x^-) = \alpha\big[f(x^+) - f(x^-)\big] = \alpha g(x),$$

and if $\alpha < 0$, then

$$g(\alpha x) = -\alpha g(-x) = -\alpha\big[g(x^- - x^+)\big] = -\alpha\big[f(x^-) - f(x^+)\big] = \alpha g(x).$$

Thus, g is a linear functional on X, which is clearly a unique extension of f.

Problem 30.2. *A vector lattice is called* **order complete** *if every nonempty subset that is bounded from above has a least upper bound (also called the supremum of the set).*

Show that if X is a vector lattice, then its order dual $X^\sim$ is an order complete vector lattice.

Solution. Let A be a nonempty subset of $X^{\sim}$ that is bounded from above by some $g \in X^{\sim}$. By replacing A with the set $\{g - f\colon f \in A\}$, we can assume that $A \subseteq X^{\sim}_{+}$. Let B denote the collection of all finite suprema of A, i.e., $f \in B$ if and only if there exist $f_1, \dots, f_n \in A$ with $f = \bigvee_{i=1}^{n} f_i$. Clearly, $f \leq g$ also holds for all $f \in B$. Next, define $h\colon X^+ \longrightarrow \mathbf{R}^+$ by

$$h(x) = \sup\{f(x)\colon f \in B\}$$

for each $x \in X^+$. Clearly, $0 \leq h(x) \leq g(x)$ holds.

Let $x, y \in X^+$. Since $f(x+y) = f(x) + f(y) \leq h(x) + h(y)$ holds for all $f \in B$, we see that

$$h(x+y) \leq h(x) + h(y).$$

On the other hand, given $\varepsilon > 0$ choose $f_1, f_2 \in B$ such that $h(x) - \varepsilon < f_1(x)$ and $h(y) - \varepsilon < f_2(y)$. Taking into account that $f_1 \vee f_2 \in B$, we see that

$$\begin{aligned} h(x) + h(y) - 2\varepsilon &\leq f_1(x) + f_2(y) \leq f_1 \vee f_2(x) + f_1 \vee f_2(y) \\ &= f_1 \vee f_2(x+y) \leq h(x+y) \end{aligned}$$

holds, for all $\varepsilon > 0$. Thus,

$$h(x) + h(y) \leq h(x+y)$$

also holds, and so $h(x+y) = h(x) + h(y)$.

By the preceding problem, h extends uniquely to a positive linear functional. Clearly, $f \leq h$ holds for all $f \in A$. On the other hand, if $f \leq \phi$ holds for all $f \in A$, then $f \leq \phi$ also holds for all $f \in B$. This easily implies $h \leq \phi$. That is, $h = \sup A$ holds in $X^{\sim}$.

Problem 30.3. *Show that the collection of all bounded functions on* $[0, 1]$ *is an ideal of* $\mathbf{R}^{[0,1]}$. *Also, show that* $C[0, 1]$ *is a vector sublattice of* $\mathbf{R}^{[0,1]}$ *but not an ideal.*

Solution. Let $f\colon [0, 1] \longrightarrow \mathbf{R}$ be a bounded function. If $|f(x)| \leq M$ holds for all $x \in [0, 1]$ and $g \in \mathbf{R}^{[0,1]}$ satisfies $|g| \leq |f|$, then $|g(x)| \leq M$ also holds for all $x \in [0, 1]$. This implies that the space of all bounded functions is an ideal of $\mathbf{R}^{[0,1]}$.

The function $\chi_{(0,\frac{1}{2})}$ is not a continuous function and satisfies $0 \leq \chi_{(0,\frac{1}{2})} \leq \mathbf{1}$, where $\mathbf{1}$ denotes the constant function one on $[0, 1]$. Hence, $C[0, 1]$ is not an ideal of $\mathbf{R}^{[0,1]}$.

Problem 30.4. *Let X be a vector lattice. Show that a norm $\|\cdot\|$ on X is a lattice norm if and only if it satisfies the following two properties:*

a. *If $0 \le x \le y$, then $\|x\| \le \|y\|$, and*
b. *$\|x\| = \| |x| \|$ holds for all $x \in X$.*

Solution. Assume that $\|\cdot\|$ is a lattice norm. Clearly, $0 \le x \le y$ implies $\|x\| \le \|y\|$. Also, $|x| = \big||x|\big|$ holds, and so $\|x\| \le \big\||x|\big\| \le \|x\|$.

Conversely, assume (a) and (b) to be true. If $|x| \le |y|$, then

$$\|x\| = \big\||x|\big\| \le \big\||y|\big\| = \|y\|$$

so that $\|\cdot\|$ is a lattice norm.

Problem 30.5. *Show that in a normed vector lattice X its positive cone X^+ is a closed set.*

Solution. From Theorem 30.1(3) we see that

$$\big|x^- - y^-\big| = \big|(-x)^+ - (-y)^+\big| \le \big|x - y\big|.$$

This implies that the function $x \longmapsto x^-$ from X into X is (uniformly) continuous. Thus, $X^+ = \{x \in X\colon x^- = 0\}$ is a closed set.

Problem 30.6. *Let X be a normed vector lattice. Assume that $\{x_n\}$ is a sequence of X such that $x_n \le x_{n+1}$ holds for all n. Show that if $\lim x_n = x$ holds in X, then the vector x is the least upper bound of the sequence $\{x_n\}$ in X. In symbols, $x_n \uparrow x$ holds.*

Solution. Assume that $\{x_n\}$ satisfies $x_n \le x_{n+1}$ for each n and $\lim x_n = x$. Then, $x_{n+p} - x_n \ge 0$ holds for all n and all p and $\lim_{p\to\infty}(x_{n+p} - x_n) = x - x_n$. Since (by Problem 30.5) the positive cone X^+ is closed, we see that $x - x_n \ge 0$, or $x \ge x_n$ for each n. This shows that x is an upper bound for the sequence $\{x_n\}$.

To see that x is the least upper bound for the sequence $\{x_n\}$, assume that $y \ge x_n$ holds for each n. So, $y - x_n \ge 0$ holds for all n and $\lim(y - x_n) = y - x$. Using once more that X^+ is closed, we get $y - x \in X^+$. That is, $y - x \ge 0$, or $y \ge x$. Therefore, $x = \sup\{x_n\}$, or $x_n \uparrow x$ holds true, as desired.

Problem 30.7. *Assume that $x_n \to x$ holds in a Banach lattice and let $\{\epsilon_n\}$ be a sequence of strictly positive real numbers. Show that there exists a subsequence $\{x_{k_n}\}$ of $\{x_n\}$ and some positive vector u such that $|x_{k_n} - x| \le \epsilon_n u$ holds for each n.*

Solution. An easy inductive argument guarantees the existence of a subsequence $\{x_{k_n}\}$ of $\{x_n\}$ satisfying $\|x_{k_n} - x\| \le \epsilon_n 2^{-n}$ for each n. Now, notice that the series of

positive vectors $\sum_{n=1}^{\infty}(\epsilon_n)^{-1}|x_{k_n}-x|$ is absolutely summable. Since X is a Banach space, $u=\sum_{n=1}^{\infty}(\epsilon_n)^{-1}|x_{k_n}-x|$ exists in X. Now, a glance at Problem 30.6 shows that $(\epsilon_n)^{-1}|x_{k_n}-x|\leq u$ for each n. Thus, $|x_{k_n}-x|\leq \epsilon_n u$ holds for each n, as desired.

Problem 30.8. *Let $T: X\to Y$ be a positive operator between two normed vector lattices, i.e, $x\geq 0$ in X implies $Tx\geq 0$ in Y. If X is a Banach lattice, then show that T is continuous.*

Solution. Let $T: X\longrightarrow Y$ be a positive operator, where X is a Banach lattice and Y is a normed vector lattice. Assume by way of contradiction that T is not continuous. Then, there exist a sequence $\{x_n\}$ of X and some $\epsilon>0$ satisfying $x_n\to 0$ and $\|Tx_n\|\geq\epsilon$ for each n. By Problem 30.7 there exists a subsequence $\{y_n\}$ of $\{x_n\}$ and some $u\in X^+$ satisfying $|y_n|\leq\frac{1}{n}u$ for each n. Now, notice that the positivity of T implies $|Ty_n|\leq T|y_n|\leq\frac{1}{n}Tu$ for each n, and so $\|Ty_n\|\leq\frac{1}{n}\|Tu\|$ for each n. Since $\frac{1}{n}\|Tu\|\to 0$, it follows that $\|Ty_n\|\to 0$ contrary to $\|Ty_n\|>\epsilon$ for each n. Thus, T is a continuous operator.

Problem 30.9. *Show that any two complete lattice norms on a vector lattice must be equivalent.*

Solution. If $\|\cdot\|_1$ and $\|\cdot\|_2$ are two complete lattice norms on a vector lattice X, then, by Problem 30.8, the identity operator $I:(X,\|\cdot\|_1)\longrightarrow(X,\|\cdot\|_2)$ is a homeomorphism. That is, $\|\cdot\|_1$ and $\|\cdot\|_2$ are two equivalent norms.

Problem 30.10. *The* **averaging operator** $A:\ell_\infty\to\ell_\infty$ *is defined by*

$$A(x)=\Big(x_1,\frac{x_1+x_2}{2},\frac{x_1+x_2+x_3}{3},\ldots,\frac{x_1+x_2+\cdots+x_n}{n},\ldots\Big)$$

for each $x=(x_1,x_2,\ldots)\in\ell_\infty$. Establish the following:

a. *A is a positive operator.*
b. *A is a continuous operator.*
c. *The vector space*

$$V=\Big\{x=(x_1,x_2,\ldots)\in\ell_\infty:\ \{\tfrac{x_1+x_2+\cdots+x_n}{n}\}\ \text{converges in } \mathbb{R}\Big\}$$

is a closed subspace of ℓ_∞. Is $V=\ell_\infty$?

Solution. (a) If $x=(x_1,x_2,\ldots)\geq 0$, then $x_i\geq 0$ for each i and so $\frac{x_1+x_2+\cdots+x_n}{n}\geq 0$ for each n. This implies $A(x)\geq 0$, and so A is a positive operator.

(b) By Problem 30.8 every positive operator on a Banach lattice is continuous. Therefore, A (as a positive operator) is continuous.

(c) We know from Problem 29.5 that the vector space of all convergent sequences

$$c = \{x = (x_1, x_2, \ldots) \in \ell_\infty\colon \lim_{n\to\infty} x_n \text{ exists in } \mathbb{R}\}$$

is a closed subspace of ℓ_∞. Clearly, $V = A^{-1}(c)$. Since A is continuous, the latter guarantees that V is a closed subspace of ℓ_∞.

There are bounded sequences having divergent sequences of averages. Here is an example:

$$(1, -1, 1, 1, -1, -1, 1, 1, 1, 1, 1, 1, -1, -1, -1, -1, -1, -1, \ldots).$$

Hence, V is a proper closed subspace of ℓ_∞.

Problem 30.11. *This problem shows that for a normed vector lattice X its norm dual X^* may be a proper ideal of its order dual $X^\sim$. Let X be the collection of all sequences $\{x_n\}$ such that $x_n = 0$ for all but a finite number of terms (depending on the sequence). Show that:*

a. *X is a function space.*
b. *X equipped with the sup norm is a normed vector lattice, but not a Banach lattice.*
c. *If $f\colon X \to \mathbb{R}$ is defined by $f(x) = \sum_{n=1}^{\infty} n x_n$ for each $x = \{x_n\} \in X$, then f is a positive linear functional on X that is not continuous.*

Solution. (a) Routine.

(b) If $\mathbf{x}_n = (1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots)$, then $\{\mathbf{x}_n\}$ is Cauchy sequence of X that does not converge in X.

(c) Clearly, f is a positive linear functional. If $\mathbf{e}_n$ denotes the sequence whose n[th] component is one and every other zero, then $\|\mathbf{e}_n\|_\infty = 1$ and $n = f(\mathbf{e}_n) \le \|f\|$. That is, $\|f\| = \infty$.

Problem 30.12. *Determine the norm completion of the normed vector lattice of the preceding problem.*

Solution. Let

$$c_0 = \left\{x = (x_1, x_2, \ldots) \in \ell_\infty\colon \lim_{n\to\infty} x_n = 0\right\}.$$

Clearly, c_0 is a vector sublattice of ℓ_∞. Also, it is not difficult to see that c_0 is a closed subspace, and so c_0 is a Banach lattice (with the sup norm). We claim that c_0 is the norm completion of the normed vector lattice X of the preceding problem.

To see this, note first that X is a vector sublattice of c_0. Now, let $x = (x_1, x_2, \dots) \in c_0$ and let $\varepsilon > 0$. Choose some n with $|x_k| < \varepsilon$ for all $k \geq n$, and note that the element $y = (x_1, \dots, x_n, 0, 0, \dots) \in X$ satisfies $\|x - y\|_\infty \leq \varepsilon$. Thus, X is dense in c_0, and our claim follows.

Problem 30.13. *Let $C_c(X)$ be the normed vector lattice—with the sup norm—of all continuous real-valued functions on a Hausdorff locally compact topological space X. Determine the norm completion of $C_c(X)$.*

Solution. Consider the vector space of functions

$$c_0(X) = \{f \in C(X) \colon \forall\, \varepsilon > 0\ \exists\, K \text{ compact with } |f(x)| < \varepsilon \text{ for } x \notin K\}.$$

Clearly, $c_0(X)$ is a vector sublattice of $B(X)$. We claim that $c_0(X)$ is a closed subspace. To see this, let $\{f_n\} \subseteq c_0(X)$ satisfy $f_n \longrightarrow f$ in $B(X)$, and let $\varepsilon > 0$. By Theorem 9.2, $f \in C(X)$. Pick some n with $\|f - f_n\|_\infty < \varepsilon$, and then select a compact set K with $|f_n(x)| < \varepsilon$ for $x \notin K$. Thus,

$$\big|f(x)\big| \leq \big|f(x) - f_n(x)\big| + \big|f_n(x)\big| < 2\varepsilon$$

holds for each $x \notin K$, and so $f \in c_0(X)$. Therefore, $c_0(X)$ (with the sup norm) is a Banach lattice.

Clearly, $C_c(X)$ is a vector sublattice of $c_0(X)$, and we claim that $C_c(X)$ is dense in $c_0(X)$. To see this, let $f \in c_0(X)$ and let $\varepsilon > 0$. Choose some compact set K with $|f(x)| < \varepsilon$ for all $x \notin K$, and then use Theorem 10.8 to pick some $g \in C_c(X)$ with $g(x) = 1$ for all $x \in K$ and $0 \leq g(x) \leq 1$ for $x \notin K$. Then, $fg \in C_c(X)$ and $\|fg - f\|_\infty \leq \varepsilon$ holds, proving that $\overline{C_c(X)} = c_0(X)$. Thus, $c_0(X)$ is the norm completion of $C_c(X)$.

Problem 30.14. *Let X and Y be two vector lattices, and let $T\colon X \to Y$ be a linear operator. Show that the following statements are equivalent:*

a. *$T(x \vee y) = T(x) \vee T(y)$ holds for all $x, y \in X$.*
b. *$T(x \wedge y) = T(x) \wedge T(y)$ holds for all $x, y \in X$.*
c. *$T(x) \wedge T(y) = 0$ holds in Y whenever $x \wedge y = 0$ holds in X.*
d. *$|T(x)| = T(|x|)$ holds for all $x \in X$.*

(A linear operator T that satisfies the preceding equivalent statements is referred to as a **lattice homomorphism**.*)*

Solution. (1)$\Longrightarrow$(2) From the identity (a) of Problem 9.1, we get

$$\begin{aligned} T(x \wedge y) &= T(x + y - x \vee y) = T(x) + T(y) - T(x \vee y) \\ &= T(x) + T(y) - T(x) \vee T(y) = T(x) \wedge T(y). \end{aligned}$$

(2) $\Longrightarrow$ (3) If $x \wedge y = 0$, then

$$T(x) \wedge T(y) = T(x \wedge y) = T(0) = 0.$$

(3) $\Longrightarrow$ (4) Using the identity (e) of Problem 9.1, we see that

$$\begin{aligned}|T(x)| &= |T(x^+) - T(x^-)| = T(x^+) \vee T(x^-) - T(x^+) \wedge T(x^-)\\ &= T(x^+) \vee T(x^-) = T(x^+) + T(x^-) - T(x^+) \wedge T(x^-)\\ &= T(x^+) + T(x^-) = T(x^+ + x^-) = T(|x|).\end{aligned}$$

(4) $\Longrightarrow$ (1) From the identity (f) of Problem 9.1, we get

$$\begin{aligned}T(x \vee y) &= T\big(\tfrac{1}{2}[x + y + |x - y|]\big) = \tfrac{1}{2}\big[T(x) + T(y) + T(|x - y|)\big]\\ &= \tfrac{1}{2}\big[T(x) + T(y) + |T(x) - T(y)|\big] = T(x) \vee T(y).\end{aligned}$$

Problem 30.15. *Let ℓ_∞ be the Banach lattice of all bounded real sequences; that is, $\ell_\infty = B(\mathbb{N})$, and let $\{r_1, r_2, \dots\}$ be an enumeration of the rational numbers of $[0, 1]$. Show that the mapping $T: C[0, 1] \to \ell_\infty$ defined by $T(f) = (f(r_1), f(r_2), \dots)$ is a lattice isometry that is not onto.*

Solution. Clearly, T is a linear operator. Let $f \in C[0, 1]$. Since f is a continuous function and the set of all rational numbers of $[0, 1]$ is a dense set, it easily follows that

$$\begin{aligned}\|T(f)\|_\infty &= \sup\{|f(r_n)|:\ n = 1, 2, \dots\}\\ &= \sup\{|f(x)|:\ x \in [0, 1]\} = \|f\|_\infty.\end{aligned}$$

In addition, note that

$$|T(f)| = \big(|f(r_1)|, |f(r_2)|, \dots\big) = \big(|f|(r_1), |f|(r_2), \dots\big) = T(|f|),$$

which shows that T is a lattice isometry.

To see that T is not onto, note that

$$T(f) \neq (0, 1, 0, 1, \dots)$$

holds for each $f \in C[0, 1]$.

Problem 30.16. *Let X be a normed vector lattice. Then show that an element $x \in X$ satisfies $x \geq 0$ if and only if $f(x) \geq 0$ holds for each continuous positive linear functional f on X.*

Solution. If $x \geq 0$ holds, then clearly $f(x) \geq 0$ also holds for each $0 \leq f \in X^\sim$.

For the converse, assume that x is fixed and satisfies $f(x) \geq 0$ for each $f \in X^\sim_+$. Let $0 \leq f \in X^\sim$ be fixed. Since $-g(x) \leq 0$ holds for all $0 \leq g \leq f$, it follows from Theorem 30.3 that

$$0 \leq f(x^-) = \sup\{-g(x)\colon\ g \in X^\sim \text{ and } 0 \leq g \leq f\} \leq 0.$$

That is, $f(x^-) = 0$ holds for all $0 \leq f \in X^\sim$, and consequently $f(x^-) = 0$ for all $f \in X^*$. From Theorem 29.4, we see that $x^- = 0$. Thus, $x = x^+ - x^- = x^+ \geq 0$, as required.

Problem 30.17. *Let X be a Banach lattice. If $0 \leq x \in X$, then show that*

$$\|x\| = \sup\{f(x)\colon\ 0 \leq f \in X^* \text{ and } \|f\| = 1\}.$$

Solution. Let $x \geq 0$. In view of the inequality $|f(x)| \leq |f|(x) \leq \|f\| \cdot \|x\|$, we have

$$\begin{aligned}\|x\| &= \sup\{|f(x)|\colon\ f \in X^* \text{ and } \|f\| = 1\}\\ &\leq \sup\{|f|(x)\colon\ f \in X^* \text{ and } \|f\| = 1\}\\ &= \sup\{f(x)\colon\ 0 \leq f \in X^* \text{ and } \|f\| = 1\} \leq \|x\|,\end{aligned}$$

and the conclusion follows.

Problem 30.18. *Assume that $\varphi\colon [0, 1] \to \mathbf{R}$ is a strictly monotone continuous function and that $T\colon C[0, 1] \to C[0, 1]$ is a continuous linear operator. If $T(\varphi f) = \varphi T(f)$ holds for each $f \in C[0, 1]$ (where φf denotes the pointwise product of φ and f). Show that there exists a unique function $h \in C[0, 1]$ satisfying $T(f) = hf$ for all $f \in C[0, 1]$.*

Solution. Taking $f = 1$, the constant one function, and letting $h = T1$, we obtain $T(\varphi) = h\varphi$, and by induction $T(\varphi^n) = h\varphi^n$ for each $n \geq 0$. Hence, by the linearity of T, we see that

$$T\big(P(\varphi)\big) = hP(\varphi) \qquad (\star)$$

for each polynomial P of one variable. Since the function φ is strictly increasing, the algebra $\mathcal{A} = \{P(\varphi)\colon\ P \text{ polynomial}\}$ separates the points and contains the constant function $\mathbf{1}$. Consequently, by the Stone–Weierstrass Theorem 11.5, $\mathcal{A}$ is dense in $C[0, 1]$. From ($\star$), it easily follows that $T(f) = hf$ for each $f \in C[0, 1]$.

Problem 30.19. *If $f \in C[0, 1]$, then the polynomials*

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} f\left(\tfrac{k}{n}\right) x^k (1-x)^{n-k},$$

where $\binom{n}{k}$ is the binomial coefficient defined by $\binom{n}{k} = \frac{n!}{k!(n-k)!}$, are known as the **Bernstein polynomials** *of f.*

Show that if $f \in C[0, 1]$, then the sequence $\{B_n\}$ of Bernstein polynomials of f converges uniformly to f.

Solution. Let $\{T_n\}$ be the sequence of positive operators from $C[0, 1]$ into $C[0, 1]$ defined by

$$T_n f(t) = \sum_{k=0}^{n} \binom{n}{k} f\left(\tfrac{k}{n}\right) t^k (1-t)^{n-k}$$

for all $f \in C[0, 1]$ and each $t \in [0, 1]$. We must show that

$$\lim \|T_n f - f\|_\infty = 0$$

holds for each $f \in C[0, 1]$. By Korovkin's Theorem 30.13, it suffices to establish that $\lim \|T_n f - f\|_\infty = 0$ holds for $f = \mathbf{1}$, x, and x^2.

To do this, we need some elementary identities. First note that by the binomial theorem

$$\sum_{k=0}^{n} \binom{n}{k} t^k (1-t)^{n-k} = \left[t + (1-t)\right]^n = 1 \tag{$\star$}$$

holds for all t. Differentiating ($\star$), we get

$$\sum_{k=0}^{n} \binom{n}{k} \left[k t^{k-1} (1-t)^{n-k} - (n-k) t^k (1-t)^{n-k-1}\right]$$

$$= \sum_{k=0}^{n} \binom{n}{k} t^{k-1} (1-t)^{n-k-1} (k - nt) = 0.$$

Multiplication by $t(1-t)$ yields

$$\sum_{k=0}^{n}\binom{n}{k}t^k(1-t)^{n-k}(k-nt)=0,$$

and by using $(\star)$, we see that

$$\sum_{k=0}^{n}\binom{n}{k}\tfrac{k}{n}\,t^k(1-t)^{n-k}=t. \qquad (\star\star)$$

Differentiating $(\star\star)$ yields

$$\sum_{k=0}^{n}\binom{n}{k}\tfrac{k}{n}\,t^{k-1}(1-t)^{n-k-1}(k-nt)=1,$$

and multiplying by $t(1-t)$, we get

$$\sum_{k=0}^{n}\binom{n}{k}\tfrac{k}{n}\,t^k(1-t)^{k-n}(k-nt)=t(1-t).$$

That is,

$$\sum_{k=0}^{n}\binom{n}{k}\left(\tfrac{k}{n}\right)^2 t^k(1-t)^{k-n}-t\sum_{k=0}^{n}\binom{n}{k}\tfrac{k}{n}\,t^k(1-t)^{k-n}=\tfrac{t(1-t)}{n},$$

and by taking into account $(\star\star)$, we see that

$$\sum_{k=0}^{n}\binom{n}{k}\left(\tfrac{k}{n}\right)^2 t^k(1-t)^{n-k}-t^2=\tfrac{t(t-1)}{n}. \qquad (\star\star\star)$$

The identities $(\star)$, $(\star\star)$, and $(\star\star\star)$ can be rewritten as follows:

$$T_n\mathbf{1}=\mathbf{1},\quad T_nx=x,\quad\text{and}\quad \left[T_nx^2-x^2\right](t)=\tfrac{t(1-t)}{n}.$$

Now, note that these identities readily imply that $\lim\|T_nf-f\|_\infty=0$ holds for $f=\mathbf{1}$, x, and x^2.

Problem 30.20. *Let $T\colon C[0,1]\to C[0,1]$ be a positive operator. Show that if $Tf=f$ holds true when f equals $\mathbf{1}$, x, and x^2, then T is the identity operator (that is, $Tf=f$ holds for each $f\in C[0,1]$).*

Solution. For each n, let $T_n = T$. Clearly, $\lim T_n f = f$ holds in $C[0,1]$ when $f = \mathbf{1}$, x, and x^2. By Korovkin's Theorem 30.13, we have $Tf = \lim T_n f = f$ for each $f \in C[0,1]$.

Problem 30.21 (Korovkin). *Let $\{T_n\}$ be a sequence of positive operators from $C[0,1]$ into $C[0,1]$ satisfying $T_n\mathbf{1} = \mathbf{1}$. If there exists some $c \in [0,1]$ such that $\lim T_n g = 0$ holds for the function $g(t) = (t-c)^2$, then show that $\lim T_n f = f(c)\cdot\mathbf{1}$ holds for all $f \in C[0,1]$.*

Solution. Let $f \in C[0,1]$ and let $\varepsilon > 0$. It suffices to show that there exist constants C_1 and C_2 such that

$$\|T_n f - f(c)\cdot\mathbf{1}\|_\infty \le \varepsilon + C_1\|T_n\mathbf{1} - \mathbf{1}\|_\infty + C_2\|T_n g\|_\infty$$

holds for all n.

Set $M = \|f\|_\infty$. By the continuity of f at the point c there exists some $\delta > 0$ such that $-\varepsilon < f(t) - f(c) < \varepsilon$ holds whenever $t \in [0,1]$ satisfies $|t-c| < \delta$. Next, observe that

$$-\varepsilon - \tfrac{2M}{\delta^2}(t-c)^2 \le f(t) - f(c) \le \varepsilon + \tfrac{2M}{\delta^2}(t-c)^2 \qquad \text{(a)}$$

holds for all $t \in [0,1]$. (To see this, repeat the arguments in the proof of Theorem 30.13.) Since each T_n is positive and linear, it follows from **(a)** that

$$-\varepsilon T_n\mathbf{1} - \tfrac{2M}{\delta^2}T_n g \le T_n f - f(c)\cdot T_n\mathbf{1} \le \varepsilon T_n\mathbf{1} + \tfrac{2M}{\delta^2}T_n g.$$

Put $C = \frac{2M}{\delta^2}$, and note that

$$|T_n f - f(c)\cdot T_n\mathbf{1}| \le \varepsilon T_n\mathbf{1} + CT_n g = \varepsilon\mathbf{1} + \varepsilon|T_n\mathbf{1} - \mathbf{1}| + CT_n g.$$

Consequently,

$$\begin{aligned} |T_n f - f(c)\cdot\mathbf{1}| &\le |T_n f - f(c)\cdot T_n\mathbf{1}| + |f(c)|\cdot|T_n\mathbf{1} - \mathbf{1}| \\ &\le \varepsilon\mathbf{1} + (\varepsilon + |f(c)|)|T_n\mathbf{1} - \mathbf{1}| + CT_n g, \end{aligned}$$

and so

$$\|T_n f - f(c)\cdot\mathbf{1}\|_\infty \le \varepsilon + (\varepsilon + |f(c)|)\|T_n\mathbf{1} - \mathbf{1}\|_\infty + C\|T_n g\|_\infty,$$

31. L_p-SPACES

Problem 31.1. *Let $f \in L_p(\mu)$, and let $\epsilon > 0$. Show that*

$$\mu^*(\{x \in X\colon\ |f(x)| \geq \epsilon\}) \leq \epsilon^{-p} \int |f|^p \, d\mu.$$

Solution. Consider the measurable set $E = \{x \in X\colon\ |f(x)| \geq \varepsilon\}$, and note that $E = \{x \in X\colon\ |f(x)|^p \geq \varepsilon^p\}$. Thus,

$$\int |f|^p \, d\mu \geq \int \chi_E |f|^p \, d\mu \geq \int \varepsilon^p \chi_E \, d\mu = \varepsilon^p \mu^*(E).$$

Problem 31.2. *Let $\{f_n\}$ be a sequence of some $L_p(\mu)$-space with $1 \leq p < \infty$. Show that if $\lim \|f_n - f\|_p = 0$ holds in $L_p(\mu)$, then $\{f_n\}$ converges in measure to f.*

Solution. From the preceding problem, we see that

$$\mu^*\big(\{x \in X\colon\ |f_n(x) - f(x)| \geq \varepsilon\}\big) \leq \varepsilon^{-p} \int \left| f_n - f \right|^p d\mu$$

holds. Clearly, this inequality shows that $f_n \xrightarrow{\mu} f$ holds whenever $\lim \|f_n - f\|_p = 0$.

Problem 31.3. *Let (X, S, μ) be a measure space and consider the set*

$$E = \left\{\chi_A\colon\ A \in \Lambda_\mu \text{ with } \mu^*(A) < \infty\right\}.$$

Show that E is a closed subset of $L_1(\mu)$ (and hence, a complete metric space in its own right). Use this conclusion and the identity

$$\mu(A \Delta B) = \int \left|\chi_A - \chi_B\right| d\mu = \|\chi_A - \chi_B\|_1$$

to provide an alternate solution to Problem 14.12(c).

Solution. Assume that $\{\chi_{A_n}\}$ is a sequence of E such that $\int \left|\chi_{A_n} - f\right| d\mu \to 0$ holds for some $f \in L_1(\mu)$. By Lemma 31.6 there exists a subsequence $\{\chi_{A_{k_n}}\}$ of $\{\chi_{A_n}\}$ such that $\chi_{A_{k_n}} \to f$ a.e. This implies (how?) that $f = \chi_A$ a.e. for some $A \in \Lambda_\mu$ with $\mu^*(A) < \infty$. Thus, $f \in E$ and so E is a closed subset of $L_1(\mu)$.

Problem 31.4. *Show that equality holds in the inequality*

$$a^t b^{1-t} \le ta + (1-t)b, \;\; 0 < t < 1; \;\; a \ge 0; \;\; b \ge 0$$

if and only if $a = b$. Use this to show that if $f \in L_p(\mu)$ and $g \in L_q(\mu)$, where $1 < p, q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, then $\int |fg|\, d\mu = \|f\|_p \cdot \|g\|_q$ holds if and only if there exist two constants C_1 and C_2 (not both zero) such that $C_1|f|^p = C_2|g|^q$ holds.

Solution. Clearly, if $a = b \ge 0$, then $a^t b^{t-1} = ta + (1-t)b = a$ holds. For the converse, let $a^t b^{1-t} = ta + (1-t)b$ hold for some $a, b > 0$. Put $y = \frac{a}{b}$, and rewrite the given equality as $1 - t + ty - y^t = 0$. Since the function $f(x) = 1 - t + tx - x^t$ for $x \ge 0$ (and some fixed $0 < t < 1$) attains its minimum when $x = 1$ (see the proof of Lemma 31.2), it follows that $y = \frac{a}{b} = 1$, and so $a = b$. Thus, $a^t b^{1-t} = ta + (1-t)b$ holds if and only if $a = b$.

For the second part, assume first that there exist two constants C_1 and C_2 (which are not both zero) such that $C_1|f|^p = C_2|g|^q$. We can assume $C_1 > 0$ and $C_2 \ge 0$. Then, we have

$$\begin{aligned}
\int |fg|\, d\mu &= \int \left(\tfrac{C_2}{C_1}\right)^{\frac{1}{p}} |g|^{\frac{q}{p}} |g|\, d\mu = \left(\tfrac{C_2}{C_1}\right)^{\frac{1}{p}} \int |g|^q\, d\mu \\
&= \left[\int \left(\tfrac{C_2}{C_1}\right) |g|^q\, d\mu\right]^{\frac{1}{p}} \cdot \left[\int |g|^q\, d\mu\right]^{\frac{1}{q}} \\
&= \left(\int |f|^p\, d\mu\right)^{\frac{1}{p}} \cdot \left(\int |g|^q\, d\mu\right)^{\frac{1}{q}} = \|f\|_p \cdot \|g\|_q.
\end{aligned}$$

For the converse, assume $\int |fg|\, d\mu = \|f\|_p \cdot \|g\|_q$. If either f or g is zero, then the conclusion is trivial. (If $f = 0$, then put $C_1 = 1$ and $C_2 = 0$.) So, we can assume $f \ne 0$ and $g \ne 0$. Taking $t = \frac{1}{p}$, $a = \left(\frac{|f(x)|}{\|f\|_p}\right)^p$, and $b = \left(\frac{|g(x)|}{\|g\|_q}\right)^q$, the inequality $a^t b^{1-t} \le ta + (1-t)b$ gives

$$0 \le \frac{1}{p}\left(\frac{|f(x)|}{\|f\|_p}\right)^p + \frac{1}{q}\left(\frac{|g(x)|}{\|g\|_q}\right)^q - \frac{|f(x)g(x)|}{\|f\|_p \cdot \|g\|_q}.$$

Integrating (and using our hypothesis), we get

$$\begin{aligned}
0 &\le \int \left[\frac{1}{p}\left(\frac{|f(x)|}{\|f\|_p}\right)^p + \frac{1}{q}\left(\frac{|g(x)|}{\|g\|_q}\right)^q - \frac{|f(x)g(x)|}{\|f\|_p \cdot \|g\|_q}\right] d\mu(x) \\
&= \frac{1}{p} + \frac{1}{q} - \frac{\int |f(x)g(x)|\, d\mu(x)}{\|f\|_p \cdot \|g\|_q} = 1 - 1 = 0.
\end{aligned}$$

Consequently,

$$\frac{|f(x)g(x)|}{\|f\|_p\cdot\|g\|_q} = \frac{1}{p}\left(\frac{|f(x)|}{\|f\|_p}\right)^p + \frac{1}{q}\left(\frac{|g(x)|}{\|g\|_q}\right)^q$$

holds for almost all x, and by the first part of the problem, we see that $\left(\frac{|f(x)|}{\|f\|_p}\right)^p = \left(\frac{|g(x)|}{\|g\|_q}\right)^q$ holds, so that

$$(\|g\|_q)^q|f(x)|^p = (\|f\|_p)^p|g(x)|^q$$

holds for almost all x, as required.

Problem 31.5. *Assume that $\mu^*(X) = 1$ and $0 < p < q \le \infty$. If f is in $L_q(\mu)$, then show that $\|f\|_p \le \|f\|_q$ holds.*

Solution. Assume $\mu^*(X) = 1$ and $0 < p < q < \infty$. Let $f \in L_q(\mu)$. From Theorem 31.14, we know that $L_q(\mu) \subseteq L_p(\mu)$, and so $f \in L_p(\mu)$.

Put $r = \frac{q}{p} > 1$, and then choose $s > 1$ so that $\frac{1}{r} + \frac{1}{s} = 1$ holds. Since $|f|^p \in L_r(\mu)$ and $\mathbf{1} \in L_s(\mu)$, it follows from Hölder's inequality that

$$\begin{aligned}(\|f\|_p)^p &= \int |f|^p\,d\mu = \int |f|^p\cdot\mathbf{1}\,d\mu \le \left(\int |f|^{pr}\,d\mu\right)^{\frac{1}{r}}\cdot\left(\int \mathbf{1}^s\,d\mu\right)^{\frac{1}{s}}\\ &= \left(\int |f|^{pr}\,d\mu\right)^{\frac{1}{r}} = \left(\int |f|^q\,d\mu\right)^{\frac{p}{q}} = (\|f\|_q)^p.\end{aligned}$$

Consequently, $\|f\|_p \le \|f\|_q$ holds.

If $q = \infty$, then

$$\|f\|_p = \left(\int |f|^p\,d\mu\right)^{\frac{1}{p}} \le \left(\int (\|f\|_\infty)^p\,d\mu\right)^{\frac{1}{p}} = \|f\|_\infty = \|f\|_q$$

also holds in this case.

Problem 31.6. *Let $f \in L_1(\mu)\cap L_\infty(\mu)$. Then show that*

a. *$f \in L_p(\mu)$ for each $1 < p < \infty$.*
b. *If $\mu^*(X) < \infty$, then $\lim_{p\to\infty}\|f\|_p = \|f\|_\infty$ holds.*

Solution. (a) If $M = \|f\|_\infty$, then the inequality

$$|f|^p = |f|^{p-1}\cdot|f| \le M^{p-1}\cdot|f|$$

shows that $f \in L_p(\mu)$ for each $1 < p < \infty$.

(b) Let $\{p_n\}$ be a sequence of positive real numbers satisfying $p_n > 1$ for each n and $\lim p_n = \infty$. From the inequality

$$\|f\|_{p_n} = \left(\int |f|^{p_n}\,d\mu\right)^{\frac{1}{p_n}} \le \|f\|_\infty [\mu^*(X)]^{\frac{1}{p_n}},$$

it follows that

$$\limsup \|f\|_{p_n} \le \|f\|_\infty.$$

Let $0 < \varepsilon < M$. Then, the measurable set

$$E = \left\{x \in X\colon\ |f(x)| \ge \|f\|_\infty - \varepsilon\right\}$$

satisfies $\mu^*(E) > 0$. From $\left(\|f\|_\infty - \varepsilon\right)^{p_n}\chi_E \le |f|^{p_n}$, we see that $\left(\|f\|_\infty - \varepsilon\right)\left[\mu^*(E)\right]^{\frac{1}{p_n}} \le \|f\|_{p_n}$, and so $\|f\|_\infty - \varepsilon \le \liminf \|f\|_{p_n}$ holds for all $0 < \varepsilon < M$. That is,

$$\|f\|_\infty \le \liminf \|f\|_{p_n}.$$

Thus, $\limsup \|f\|_{p_n} \le \|f\|_\infty \le \liminf \|f\|_{p_n}$ holds. This shows that $\lim \|f\|_{p_n} = \|f\|_\infty$, and from this it follows that $\lim_{p\to\infty} \|f\|_p = \|f\|_\infty$.

Problem 31.7. *Let $f \in L_2[0,1]$ satisfy $\|f\|_2 = 1$ and $\int_0^1 f(x)\,d\lambda(x) \ge \alpha > 0$. Also, for each $\beta \in \mathbb{R}$ let $E_\beta = \{x \in [0,1]\colon\ f(x) \ge \beta\}$. If $0 < \beta < \alpha$, show that*

$$\lambda(E_\beta) \ge (\beta - \alpha)^2.$$

(This inequality is known in the literature as the **Paley–Zygmund Lemma**.*)*

Solution. Assume $f \in L_2[0,1]$ satisfies the stated properties and let $0 < \beta < \alpha$. Then, note that

$$f - \beta \le (f - \beta)\chi_{E_\beta} \le f\chi_{E_\beta},$$

and so, from Hölder's inequality, it follows that

$$\begin{aligned} 0 < \alpha - \beta \le \int_0^1 f(x)\,d\lambda(x) - \beta &= \int_0^1 [f(x) - \beta]\,d\lambda(x) \le \int_0^1 f(x)\chi_{E_\beta}(x)\,d\lambda(x) \\ &\le \|f\|_2 \cdot [\lambda(E_\beta)]^{\frac{1}{2}} = [\lambda(E_\beta)]^{\frac{1}{2}}. \end{aligned}$$

This implies $\lambda(E_\beta) \ge (\alpha - \beta)^2$.

Problem 31.8. *Show that for $1 \le p < \infty$ each ℓ_p is a separable Banach lattice.*

Solution. Let e_n denote the sequence whose n[th] component is one and every other is zero. Also, denote by E the set of all finite linear combinations of $\{e_1, e_2, \ldots\}$ with rational coefficients. Clearly, E is a countable set, which we claim is also dense in ℓ_p whenever $1 \le p < \infty$.

To see this, let $x = (x_1, x_2, \ldots) \in \ell_p$ $(1 \le p < \infty)$, and let $\varepsilon > 0$. Fix some natural number n with $\sum_{i=n+1}^{\infty} |x_i|^p < \frac{\varepsilon^p}{2}$. Then, pick rational numbers $r_1, r_2, \ldots, r_n$ with $\sum_{i=1}^{n} |x_i - r_i|^p < \frac{\varepsilon^p}{2}$, and note that the element $a = (r_1, r_2, \ldots, r_n, 0, 0, \ldots) = r_1e_1 + r_2e_2 + \cdots + r_ne_n \in E$ satisfies

$$\|x - a\|_p = \Big(\sum_{i=1}^{n} |x_i - r_i|^p + \sum_{i=n+1}^{\infty} |x_i|^p\Big)^{\frac{1}{p}} < \Big(\frac{\varepsilon^p}{2} + \frac{\varepsilon^p}{2}\Big)^{\frac{1}{p}} = \varepsilon.$$

That is, the countable set E is dense in ℓ_p, and so each ℓ_p $(1 \le p < \infty)$ is a separable Banach lattice.

Problem 31.9. *Show that ℓ_∞ is not separable.*

Solution. Let $E = \{x_1, x_2, \ldots\}$ be a countable subset of ℓ_∞. Write $x_n = (x_1^n, x_2^n, \ldots)$ for each n. Now, define

$$y_n = \begin{cases} 0 & \text{if } |x_n^n| \ge 1 \\ 2 & \text{if } |x_n^n| < 1 \,. \end{cases}$$

Clearly, $y = (y_1, y_2, \ldots) \in \ell_\infty$ and

$$\|y - x_n\|_\infty \ge |y_n - x_n^n| \ge \big|y_n - |x_n^n|\big| \ge 1$$

holds for each n. Thus, $B(y, 1) \cap E = \emptyset$, and this shows that no countable subset of ℓ_∞ can be dense.

An alternate way of proving that ℓ_∞ is not separable is as follows: Consider the set F of all sequences whose coordinates are zero or one. By Problem 2.8, the set F is uncountable, and it is not difficult to see that $\|x - y\|_\infty = 1$ holds for each pair $x, y \in F$ with $x \neq y$. It follows that $\{B(x, 1): x \in F\}$ is an uncountable collection of pairwise disjoint open balls. This easily implies that every dense subset of ℓ_∞ must be uncountable.

Problem 31.10. *Show that $L_\infty([0, 1])$ (with the Lebesgue measure) is not separable.*

Solution. Write $f_x = \chi_{[0,x]}$, $0 < x < 1$. Since $\|f_x - f_y\|_\infty = 1$ holds whenever $x \neq y$, it follows that $\{B(f_x, 1)\colon x \in (0, 1)\}$ is an uncountable collection of pairwise disjoint open balls of $L_\infty([0, 1])$. This easily implies that every dense subset of $L_\infty([0, 1])$ must be uncountable, and so $L_\infty([0, 1])$ is not a separable Banach lattice.

Problem 31.11. *Let X be a Hausdorff locally compact topological space, and fix a point $a \in X$. Let μ be the measure on X defined on all subsets of X by $\mu(A) = 1$ if $a \in A$ and $\mu(A) = 0$ if $a \notin A$. In other words, μ is the Dirac measure (see Example 13.4). Show that μ is a regular Borel measure and that* $\operatorname{Supp}\mu = \{a\}$.

Solution. The regularity of μ will be established first.
1) Clearly, $\mu(A) \le 1$ holds for each $A \subseteq X$.
2) Let $B \subseteq X$. If $a \in B$, then

$$1 = \mu(B) \le \inf\{\mu(\mathcal{O})\colon \mathcal{O} \text{ open and } B \subseteq \mathcal{O}\} \le \mu(X) = 1.$$

On the other hand, if $a \notin B$, then use the open set $X \setminus \{a\}$ to see that

$$0 = \mu(B) \le \inf\{\mu(\mathcal{O})\colon \mathcal{O} \text{ open and } B \subseteq \mathcal{O}\} \le \mu\big(X \setminus \{a\}\big) = 0.$$

3) Let $B \subseteq X$. If $a \notin B$, then each subset C of B satisfies $\mu(C) = 0$, and so

$$0 = \sup\{\mu(K)\colon K \text{ compact and } K \subseteq B\} \le \mu(B) = 0.$$

Now, if $a \in B$, then using that $\{a\}$ is a compact subset of B we see that

$$1 = \mu\big(\{a\}\big) \le \sup\{\mu(K)\colon K \text{ compact and } K \subseteq B\} \le \mu(B) = 1.$$

Thus, μ is a regular Borel measure. Since $\mu\big(X \setminus \{a\}\big) = 0$, it is easy to see that $\operatorname{Supp}\mu = \{a\}$ holds.

Problem 31.12. *If $g \in C^1[a, b]$ and $f \in L_1[a, b]$, then*

a. *show that the function $F\colon [a, b] \to \mathbf{R}$ defined by $F(x) = \int_a^x f(t)\,d\lambda(t)$ is uniformly continuous, and*
b. *establish the following "Integration by Parts" formula:*

$$\int_a^b g(x)f(x)\,d\lambda(x) = g(x)F(x)\Big|_a^b - \int_a^b g'(x)F(x)\,dx.$$

Solution. (a) The uniform continuity of F follows immediately from Problem 22.6.

(b) Start by choosing some constant $C > 0$ such that $|g(x)| \le C$ and $|g'(x)| \le C$ hold for each $x \in [a, b]$. Now, by Theorem 25.3 there exists a sequence of continuous functions $\{f_n\}$ satisfying $\lim \int_a^b |f - f_n| \, d\lambda = 0$. From Lemma 31.6, we can suppose (by passing to a subsequence if necessary) that there exists some function $0 \le h \in L_1[a, b]$ satisfying $|f_n| \le h$ a.e. for each n and $f_n \to f$ a.e. Let $F_n(x) = \int_a^x f_n(t) \, dt$, and note that by the "standard" Integration by Parts Formula we have

$$\int_a^b g(x) f_n(x) \, d\lambda(x) = \int_a^b g(x) f_n(x) \, dx = g(x) F_n(x) \Big|_a^b - \int_a^b g'(x) F_n(x) \, dx. \qquad (\star)$$

From $|g f_n| \le Ch \in L_1[a, b]$, $g f_n \to g f$ a.e., and the Lebesgue Dominated Convergence Theorem, it follows that

$$\lim_{n\to\infty} \int_a^b g(x) f_n(x) \, dx = \int_a^b g(x) f(x) \, d\lambda(x).$$

Likewise, the Lebesgue Dominated Convergence Theorem implies

$$F_n(x) = \int_a^x f_n(t) \, dt \longrightarrow \int_a^x f(t) \, d\lambda(t) = F(x).$$

for each $x \in [a, b]$. Observing that $|g' F_n| \le C \int_a^b h \, d\lambda$ and $g' F_n \to g' F$, the Lebesgue Dominated Convergence Theorem once more yields

$$\lim_{n\to\infty} \int_a^b g'(x) F_n(x) \, dx = \int_a^b g'(x) F(x) \, dx.$$

Finally, letting $n \to \infty$ in $(\star)$, we obtain

$$\int_a^b g(x) f(x) \, d\lambda(x) = g(x) F(x) \Big|_a^b - \int_a^b g'(x) F(x) \, dx,$$

as desired.

Problem 31.13. *Let μ be a regular Borel measure on $\mathbf{R}^n$. Then show that the collection of all real-valued functions on $\mathbf{R}^n$ that are infinitely many times differentiable is norm dense in $L_p(\mu)$ for each $1 \le p < \infty$.*

Solution. Let $\mathcal{S}$ be the semiring consisting of the sets of the form $\prod_{i=1}^{n}[a_i, b_i)$. By Theorem 15.10, the outer measure generated by $(\mathbf{R}^n, \mathcal{S}, \mu)$ agrees with μ on the σ-algebra $\mathcal{B}$ of all Borel sets of $\mathbf{R}^n$. Thus, what needs to be shown is that given $I = \prod_{i=1}^{n}[a_i, b_i)$ and $\varepsilon > 0$, there exists some C^∞-function f with compact support such that $\|\chi_I - f\|_p < \varepsilon$.

To this end, let $I = \prod_{i=1}^{n}[a_i, b_i)$ and let $\varepsilon > 0$. The arguments of the first part of the solution of Problem 25.6 show that there is a C^∞-function $f: \mathbf{R}^n \longrightarrow [0, 1]$ satisfying $\int |\chi_I - f|\,d\mu < 2^{-p}\varepsilon^p$. Since $|\chi_I - f| \le 2$ holds, it follows that

$$\begin{aligned}\|\chi_I - f\|_p &= \Big(\int |\chi_I - f|^p\,d\mu\Big)^{\frac{1}{p}} = \Big(\int |\chi_I - f|^{p-1}\cdot|\chi_I - f|\,d\mu\Big)^{\frac{1}{p}}\\ &\le 2\Big(\int |\chi_I - f|\,d\mu\Big)^{\frac{1}{p}} < 2\cdot 2^{-1}\varepsilon = \varepsilon,\end{aligned}$$

Problem 31.14. *Let $(X, \mathcal{S}, \mu)$ be a measure space with $\mu^*(X) = 1$. Assume that a function $f \in L_1(\mu)$ satisfies $f(x) \ge M > 0$ for almost all x. Then show that $\ln(f) \in L_1(\mu)$ and that $\int \ln(f)\,d\mu \le \ln(\int f\,d\mu)$ holds.*

Solution. The function $g(t) = t - 1 - \ln t$, $t > 0$, attains its minimum value at $t = 1$. Thus, $0 = g(1) \le g(t) = t - 1 - \ln t$ holds for all $t > 0$, and so $\ln t \le t - 1$. Replacing t by $\frac{1}{t}$, the last inequality yields $1 - \frac{1}{t} \le \ln t$. Therefore,

$$1 - \tfrac{1}{t} \le \ln t \le t - 1 \qquad (\star)$$

holds for each $t > 0$.

Since the function $\ln x$ is continuous on $(0, \infty)$ and f is a measurable function, it follows that $\ln(f)$ is a measurable function. (See the solution of Problem 16.8.) Replacing t by $\frac{f(x)}{\|f\|_1}$ in $(\star)$, we see that

$$1 - \tfrac{\|f\|_1}{f(x)} \le \ln\big(f(x)\big) - \ln\big(\|f\|_1\big) \le \tfrac{f(x)}{\|f\|_1} - 1 \qquad (\star\star)$$

holds for almost all x. From our assumptions, it is easy to see that both functions $1 - \frac{\|f\|_1}{f(x)}$ and $\frac{f(x)}{\|f\|_1}$ are integrable. Thus, from $(\star\star)$ and Theorem 22.6, it follows that $\ln(f) \in L_1(\mu)$.

Finally, integrating the right inequality of $(\star\star)$ (and taking into account that $\mu^*(X) = 1$), we see that

$$\int \ln(f)\,d\mu - \ln\big(\|f\|_1\big) \le \int \tfrac{f}{\|f\|_1}\,d\mu - 1 = 0.$$

That is,

$$\int \ln(f)\,d\mu \le \ln(\|f\|_1) = \ln\left(\int f\,d\mu\right)$$

holds, as required.

Problem 31.15. Theorem 31.7 states that: *If* $1 \le p < \infty$, f *in* $L_p(\mu)$, $\{f_n\} \subseteq L_p(\mu)$, $f_n \longrightarrow f$ *a.e., and* $\lim \|f_n\|_p = \|f\|_p$, *then* $\lim \|f_n - f\|_p = 0$.

Show with an example that this theorem is false when $p = \infty$.

Solution. Consider the sequence $\{f_n\}$ of $L_\infty([0,1])$ defined by $f_n = \chi_{(\frac{1}{n},1]}$. Then $f_n \longrightarrow \mathbf{1}$ a.e., and $\|f_n\|_\infty = 1 \to 1 = \|\mathbf{1}\|_\infty$. However, $\|f_n - \mathbf{1}\|_\infty = 1$ holds for each n.

Problem 31.16. *This exercise presents a necessary and sufficient condition for the mapping* $g \longmapsto F_g$ *from* $L_\infty(\mu)$ *into* $L_1^*(\mu)$ *(defined by* $F_g(f) = \int fg\,d\mu$*) to be an isometry.*

a. *Show that for each* $g \in L_\infty(\mu)$ *the linear functional* $F_g(f) = \int fg\,d\mu$, *for* $f \in L_1(\mu)$, *is a bounded linear functional on* $L_1(\mu)$ *such that* $\|F_g\| \le \|g\|_\infty$ *holds.*
b. *Consider a nonempty set* X *and* μ *the measure defined on every subset of* X *by* $\mu(\emptyset) = 0$ *and* $\mu(A) = \infty$ *if* $A \ne \emptyset$. *Then show that* $L_1(\mu) = \{0\}$ *and* $L_\infty(\mu) = B(X)$ *[the bounded functions on* X*] and conclude from this that* $g \in L_\infty(\mu)$ *satisfies* $\|F_g\| = \|g\|_\infty$ *if and only if* $g = 0$.
c. *Let us say that a measure space* (X, S, μ) *has the* **finite subset property** *whenever every measurable set of infinite measure has a measurable subset of finite positive measure.*

   Show that the linear mapping $g \longmapsto F_g$ *from* $L_\infty(\mu)$ *into* $L_1^*(\mu)$ *is a lattice isometry if and only if* (X, S, μ) *has the finite subset property.*

Solution. (a) Let $g \in L_\infty(\mu)$. Then, for each $f \in L_1(\mu)$, we have $|fg| \le \|g\|_\infty \cdot |f|$, and so

$$|F_g(f)| = \left|\int fg\,d\mu\right| \le \int |fg|\,d\mu \le \|g\|_\infty \int |f|\,d\mu = \|g\|_\infty \cdot \|f\|_1.$$

That is, F_g is a bounded linear functional on $L_1(\mu)$, and $\|F_g\| \le \|g\|_\infty$ holds.
(b) Since every nonempty set has infinite measure, it is easy to see that there is only one step function. Namely, the constant function zero. That is, $L_1(\mu) = \{0\}$ holds. On the other hand, since every one-point set has infinite measure, each equivalence class of $L_\infty(\mu)$ consists precisely of one function. This implies that $L_\infty(\mu) = B(X)$.

Finally, note that in view of $L_1^*(\mu) = \{0\}$, we must have $F_g = 0$ for each $g \in L_\infty(\mu)$. Thus, $\|F_g\| = \|g\|_\infty$ holds if and only if $\|g\|_\infty = 0$ (i.e., if and only if $g = 0$).

(c) Assume that a measure space $(X, \mathcal{S}, \mu)$ has the finite subset property. Let $0 < g \in L_\infty(\mu)$ and let $0 < \varepsilon < \|g\|_\infty$. The set

$$E = \{x \in X\colon\ |g(x)| \geq \|g\|_\infty - \varepsilon\}$$

is measurable and $\mu^*(E) > 0$ holds. By the finite subset property, there exists a measurable set F with $F \subseteq E$ and $0 < \mu^*(F) < \infty$. Put $f = \frac{\operatorname{Sgn} g \cdot \chi_F}{\mu^*(F)} \in L_1(\mu)$, and note that $\|f\|_1 = 1$. Therefore,

$$\|F_g\| \geq |F_g(f)| = \left|\int fg\,d\mu\right| = \int_F \left[\tfrac{|g|}{\mu^*(F)}\right] d\mu \geq \|g\|_\infty - \varepsilon.$$

Since $0 < \varepsilon < \|g\|_\infty$ is arbitrary, $\|F_g\| \geq \|g\|_\infty$ holds. Now, using part (a), we see that $\|F_g\| = \|g\|_\infty$ holds for all $g \in L_\infty(\mu)$. Therefore, $g \longmapsto F_g$ is a lattice isometry.

For the converse, assume that $g \longmapsto F_g$ is a lattice isometry, and let E be a measurable set with $\mu^*(E) = \infty$. Then $g = \chi_E \in L_\infty(\mu)$, and so $\|F_g\| = \|g\|_\infty = 1$. Pick some $0 \leq f \in L_1(\mu)$ with $F_g(f) = \int fg\,d\mu = \int_E f\,d\mu > \frac{1}{2}$. It is easy to see that there exists a step function $0 \leq \phi \leq f\chi_E$ with $\int \phi\,d\mu > \frac{1}{2}$. From this, it easily follows that there exists a measurable set $F \subseteq E$ with $0 < \mu^*(F) < \infty$.

Problem 31.17. *Let $(X, \mathcal{S}, \mu)$ be a measure space. Assume that there exist measurable sets $E_1, \ldots, E_n$ such that $0 < \mu(E_i) < \infty$ for $1 \leq i \leq n$, $X = \bigcup_{i=1}^n E_i$, and each E_i does not contain any proper nonempty measurable set. Then show that $L_\infty^*(\mu) = L_1(\mu)$; that is, show that $g \mapsto F_g$ from $L_1(\mu)$ to $L_\infty^*(\mu)$ is onto.*

Solution. From our assumptions, we see that $E_i \cap E_j = \emptyset$ holds whenever $i \neq j$. For each $1 \leq i \leq n$ fix some $x_i \in E_i$ and note that $\mu^*(\{x_i\}) > 0$. If f is a measurable function and $\alpha_i = f(x_i)$, then the set $f^{-1}(\{\alpha_i\}) \cap E_i$ is nonempty and measurable. Thus, by our hypothesis, $f^{-1}(\{\alpha_i\}) \cap E_i = E_i$ holds, and therefore, f must be constant on each E_i. In other words, $f = \sum_{i=1}^n f(x_i)\chi_{E_i}$ holds for each measurable function f.

To see that $g \longmapsto F_g$ from $L_1(\mu)$ to $L_\infty^*(\mu)$ is onto, let F be an arbitrary functional in $L_\infty^*(\mu)$. Put $c_i = F(\chi_{E_i})$ for $1 \leq i \leq n$, and then let $g =$

$\sum_{i=1}^{n}\left[\frac{c_i}{\mu^*(E_i)}\right]\chi_{E_i} \in L_1(\mu)$. Note that

$$F_g(\chi_{E_i}) = \int \chi_{E_i} g\, d\mu = \int \left[\tfrac{c_i}{\mu^*(E_i)}\right]\chi_{E_i}\, d\mu = c_i = F(\chi_{E_i}).$$

Consequently,

$$\begin{aligned} F_g(f) &= F_g\Big(\sum_{i=1}^{n} f(x_i)\chi_{E_i}\Big) = \sum_{i=1}^{n} f(x_i)F_g(\chi_{E_i}) \\ &= \sum_{i=1}^{n} f(x_i)F(\chi_{E_i}) = F\Big(\sum_{i=1}^{n} f(x_i)\chi_{E_i}\Big) = F(f) \end{aligned}$$

holds for all $f \in L_1(\mu)$, and so $F = F_g$. That is, $g \longmapsto F_g$ is onto.

Problem 31.18. *Let (X, S, μ) be a measure space, and let $0 < p < 1$.*

a. *Show by a counterexample that $\|\cdot\|_p$ is no longer a norm on $L_p(\mu)$.*
b. *For each $f, g \in L_p(\mu)$ let $d(f, g) = \int |f - g|^p\, d\mu = (\|f - g\|_p)^p$. Show that d is a metric on $L_p(\mu)$ and that $L_p(\mu)$ equipped with d is a complete metric space.*

Solution. (a) Let $0 < p < 1$ and consider the space $L_p([0, 1])$. Take $f = \chi_{(0,\frac{1}{2})}$ and $g = \chi_{(\frac{1}{2},1)}$, and note that

$$\|f + g\|_p = 1 > 2^{1-\frac{1}{p}} = \left(\tfrac{1}{2}\right)^{\frac{1}{p}} + \left(\tfrac{1}{2}\right)^{\frac{1}{p}} = \|f\|_p + \|g\|_p.$$

That is, $\|\cdot\|_p$ does not satisfy the triangle inequality.
(b) If $a > 0$ and $b > 0$ (and $0 < p < 1$), then

$$\begin{aligned} (a+b)^p &= (a+b)(a+b)^{p-1} = a(a+b)^{p-1} + b(a+b)^{p-1} \\ &\le a \cdot a^{p-1} + b \cdot b^{p-1} = a^p + b^p. \end{aligned}$$

Thus, $(a + b)^p \le a^p + b^p$ holds for each $a \ge 0$ and each $b \ge 0$. This inequality easily implies that $d(f, g) = \int |f - g|^p\, d\mu$ is a metric on $L_p(\mu)$.

For the completeness, let $\{f_n\}$ be a Cauchy sequence in the metric space $(L_p(\mu), d)$, where $0 < p < 1$. By passing to a subsequence, we can assume that $\int |f_{n+1} - f_n|^p\, d\mu < 2^{-n}$ holds for each n. We shall establish the existence of some $f \in L_p(\mu)$ such that $\lim \|f_n - f\|_p = 0$.

Set $g_1 = 0$ and $g_n = |f_1| + |f_2 - f_1| + \cdots + |f_n - f_{n-1}|$ for $n \geq 2$. Clearly, $0 \leq g_n \uparrow$ and

$$\int (g_n)^p \, d\mu \leq \int |f_1|^p \, d\mu + \sum_{i=2}^{n} \int |f_i - f_{i-1}|^p \, d\mu \leq \int |f_1|^p \, d\mu + 1 < \infty$$

holds for each n. By Levi's Theorem 22.8, there exists some $g \in L_p(\mu)$ such that $0 \leq g_n \uparrow g$ a.e. From

$$|f_{n+k} - f_n| = \Big| \sum_{i=n+1}^{n+k} (f_i - f_{i-1}) \Big| \leq \sum_{i=n+1}^{n+k} |f_i - f_{i-1}| = g_{n+k} - g_n,$$

it follows that $\{f_n\}$ converges pointwise (a.e.) to some function f. Since $|f_n| = \big|f_1 + \sum_{i=2}^{n}(f_i - f_{i-1})\big| \leq g_n \leq g$ hold a.e., we see that $|f| \leq g$ a.e. also holds. Therefore, $f \in L_p(\mu)$. Now, note that $|f_n - f| \leq 2g$ and $|f_n - f|^p \longrightarrow 0$ hold, and so by the Lebesgue Dominated Convergence Theorem, we see that

$$d(f_n, f) = \int |f_n - f|^p \, d\mu \longrightarrow 0.$$

Therefore, $\big(L_p(\mu), d\big)$ is a complete metric space.

Problem 31.19. *If $(X, \mathcal{S}, \mu)$ is a finite measure space, then show that the vector space of all step functions is norm dense in $L_\infty(\mu)$.*

Solution. Let $f \in L_\infty(\mu)$ and let $\varepsilon > 0$. Choose some $C > 0$ such that $|f(x)| < C$ holds for almost all x, and then pick a partition $-C = a_0 < a_1 < \cdots < a_n = C$ of $[-C, C]$ with $a_i - a_{i-1} < \varepsilon$ for each $1 \leq i \leq n$. Let $E_i = f^{-1}\big([a_{i-1}, a_i)\big)$, and note that (since $\mu^*(X) < \infty$) the simple function $\phi = \sum_{i=1}^{n} a_i \chi_{E_i}$ is a step function satisfying $\|f - \phi\|_\infty \leq \varepsilon$.

Problem 31.20. *If K is a compact subset of a metric space X, then show that there exists a regular Borel measure μ on X such that* $\operatorname{Supp}\mu = K$.

Solution. Let K be a compact subset of a metric space X. Pick a countable dense subset $\{x_1, x_2, \ldots\}$ of K (see Problem 7.2) and then for each n consider the Dirac measure δ_{x_n} supported at the point x_n (see Example 13.4). Now, consider the measure $\mu \colon \mathcal{P}(X) \longrightarrow [0, 1]$ defined by

$$\mu(A) = \sum_{n=1}^{\infty} 2^{-n} \delta_{x_n}(A) = \sum_{n \in \hat{A}} 2^{-n},$$

where $\hat{A} = \{n \in \mathbb{N} \colon x_n \in A\}$.

Clearly, $\mu(X \setminus K) = 0$. On the other hand, if $\mathcal{O}$ is an open subset of X satisfying $\mathcal{O} \cap K \neq \emptyset$, then for some n we have $x_n \in \mathcal{O}$, and so $\mu(\mathcal{O} \cap K) \geq 2^{-n}\delta_{x_n}(\mathcal{O} \cap K) = 2^{-n} > 0$.

It remains to be shown that μ is a regular Borel measure. To this end, let $A \subseteq X$ be fixed. Note first that if $C_n = \{x_1, \ldots, x_n\} \cap A \subseteq A$, then C_n is a finite set (and hence, a compact set) and, moreover, $\mu(C_n) \uparrow \mu(A)$ holds. Therefore,

$$\mu(A) = \sup\{\mu(C) \colon\ C \text{ compact and } C \subseteq A\}.$$

In the other direction, note that if for each n we consider the open set

$$\mathcal{O}_n = X \setminus \{x_i \colon\ 1 \leq i \leq n \text{ and } x_i \notin A\},$$

then $A \subseteq \mathcal{O}_n$ and $\mu(\mathcal{O}_n) \downarrow \mu(A)$ (why?). Therefore,

$$\mu(A) = \inf\{\mu(\mathcal{O}) \colon\ \mathcal{O} \text{ open and } A \subseteq \mathcal{O}\}$$

also holds, proving that μ is a regular Borel measure.

Problem 31.21. *If $\{f_n\}$ is a norm bounded sequence of $L_2(\mu)$, then show that $f_n/n \to 0$ a.e.*

Solution. Assume that a sequence $\{f_n\} \subseteq L_2(\mu)$ satisfies $\int (f_n)^2 d\mu \leq C$ for all n, where $C > 0$ is a constant. Then,

$$\sum_{n=1}^{\infty} \int \left(\tfrac{f_n}{n}\right)^2 d\mu \leq C \sum_{n=1}^{\infty} \tfrac{1}{n^2} < \infty$$

holds. By the series version of Levi's Theorem 22.9, we know that the series $\sum_{n=1}^{\infty} \left(\frac{f_n}{n}\right)^2$ defines an integrable function. Therefore, $\frac{f_n}{n} \longrightarrow 0$ a.e. must hold.

Problem 31.22. *Let $(X, \mathcal{S}, \mu)$ be a measure space such that $\mu^*(X) = 1$. If $f, g \in L_1(\mu)$ are two positive functions satisfying $f(x)g(x) \geq 1$ for almost all x, then show that*

$$\left(\int f\, d\mu\right) \cdot \left(\int g\, d\mu\right) \geq 1.$$

Solution. Note that the functions $\sqrt{f}$ and $\sqrt{g}$ both belong to $L_2(\mu)$ and satisfy $\sqrt{f(x)}\sqrt{g(x)} \geq 1$ for almost all x. Applying Hölder's inequality, we

see that

$$1=\int 1\,d\mu\le\int\sqrt{f}\sqrt{g}\,d\mu\le\Big(\int(\sqrt{f})^2\,d\mu\Big)^{\frac{1}{2}}\cdot\Big(\int(\sqrt{g})^2\,d\mu\Big)^{\frac{1}{2}}.$$

Squaring, we get $(\int f\,d\mu)\cdot(\int g\,d\mu)\ge 1$.

Problem 31.23. *Consider a measure space $(X,\mathcal{S},\mu)$ with $\mu^*(X)=1$, and let $f,g\in L_2(\mu)$. If $\int f\,d\mu=0$, then show that*

$$\Big(\int fg\,d\mu\Big)^2\le\Big[\int g^2\,d\mu-\Big(\int g\,d\mu\Big)^2\Big]\int f^2\,d\mu.$$

Solution. Put $\alpha=\int g\,d\mu$. Then, using Hölder's inequality, we get

$$\begin{aligned}\Big|\int fg\,d\mu\Big| &= \Big|\int(fg-\alpha f)\,d\mu\Big|=\Big|\int f(g-\alpha)\,d\mu\Big|\\ &\le\int|f||g-\alpha|\,d\mu\le\Big(\int f^2\,d\mu\Big)^{\frac{1}{2}}\cdot\Big(\int(g-\alpha)^2\Big)^{\frac{1}{2}}\\ &=\Big(\int f^2\,d\mu\Big)^{\frac{1}{2}}\Big(\int g^2\,d\mu-2\alpha\int g\,d\mu+\alpha^2\Big)^{\frac{1}{2}}\\ &=\Big(\int f^2 d\mu\Big)^{\frac{1}{2}}\Big[\int g^2 d\mu-2\Big(\int g d\mu\Big)\Big(\int g d\mu\Big)+\Big(\int g d\mu\Big)^2\Big]^{\frac{1}{2}}\\ &=\Big(\int f^2\,d\mu\Big)^{\frac{1}{2}}\Big[\int g^2\,d\mu-\Big(\int g\,d\mu\Big)^2\Big]^{\frac{1}{2}},\end{aligned}$$

and our inequality follows.

Problem 31.24. *If two functions $f,g\in L_3(\mu)$ satisfy*

$$\|f\|_3=\|g\|_3=\int f^2g\,d\mu=1,$$

then show that $g=|f|$ a.e.

Solution. Let $p=\frac{3}{2}$ and $q=3$, and note that $\frac{1}{p}+\frac{1}{q}=1$. Clearly, $f^2\in L_p(\mu)=L_{\frac{3}{2}}(\mu)$, and since $g\in L_3(\mu)$, we see that $f^2g\in L_1(\mu)$.

Now, using Hölder's inequality, we obtain

$$1 = \left|\int f^2 g\,d\mu\right| \le \int f^2|g|\,d\mu \le \|f^2\|_p \cdot \|g\|_q$$
$$= \left[\int (f^2)^{\frac{3}{2}}\,d\mu\right]^{\frac{2}{3}} \cdot \|g\|_3 = (\|f\|_3)^2 \cdot \|g\|_3 = 1,$$

and so $\int f^2|g|\,d\mu = \|f^2\|_p \cdot \|g\|_q = 1$. By Problem 31.4, there exists a constant $C > 0$ such that $C|f^2|^p = |g|^q$, or $C|f|^3 = |g|^3$. From $\|f\|_3 = \|g\|_3 = 1$, we infer that $C = 1$, and so $|f|^3 = |g|^3$ holds. Therefore,

$$bigl|f| = |g| \ a.e. \qquad (\star)$$

From the relation

$$\int f^2(|g| - g)\,d\mu = \int f^2|g|\,d\mu - \int f^2 g\,d\mu = \int |f|^3\,d\mu - 1 = 1 - 1 = 0$$

and $f^2(|g| - g) \ge 0$ a.e., we conclude that $f^2(|g| - g) = 0$ a.e. Taking into account $(\star)$, the latter easily implies that $g = |g| = |f|$ a.e. holds.

Problem 31.25. *For a function $f \in L_1(\mu) \cap L_2(\mu)$ establish the following properties:*

a. *$f \in L_p(\mu)$ for each $1 \le p \le 2$, and*
b. $\lim_{p\to 1^+} \|f\|_p = \|f\|_1$.

Solution. Let $f \in L_1(\mu) \cap L_2(\mu)$; we can assume that $f(x) \in \mathbf{R}$ for each $x \in X$. Consider the measurable set $A = \{x \in X\colon |f(x)| \ge 1\}$ and then define the function $g\colon X \longrightarrow \mathbf{R}$ by

$$g(x) = \begin{cases} |f(x)|^2 & \text{if } x \in A \\ |f(x)| & \text{if } x \notin A\,, \end{cases}$$

i.e., $g = f^2\chi_A + f\chi_{A^c}$. From our hypothesis, we see that $g \in L_1(\mu)$.

(a) Let $1 \le p \le 2$. Then, the inequality

$$|f(x)|^p \le \begin{cases} |f(x)|^2 & \text{if } x \in A \\ |f(x)| & \text{if } x \notin A \end{cases} = g(x), \qquad (\star)$$

implies $f \in L_p(\mu)$ for each $1 \le p \le 2$.

(b) Let a sequence $\{p_n\}$ of the interval $[1,2]$ satisfy $p_n \longrightarrow 1$. From ($\star$), we see that $|f|^{p_n} \le g$ holds for each n. Now, from $|f|^{p_n} \longrightarrow |f|$ a.e. and the Lebesgue Dominated Convergence Theorem, we infer that

$$\lim_{n\to\infty} \|f\|_{p_n} = \lim_{n\to\infty}\Big(\int |f|^{p_n}\,d\mu\Big)^{\frac{1}{p_n}} = \int |f|\,d\mu = \|f\|_1.$$

The preceding easily implies that $\lim\limits_{p\to 1^+} \|f\|_p = \|f\|_1$ holds.

Problem 31.26. *Assume that the positive real numbers $\alpha_1, \ldots, \alpha_n$ satisfy $0 < \alpha_i < 1$ for each i and $\sum_{i=1}^n \alpha_i = 1$. If $f_1, \ldots, f_n$ are positive integrable functions on some measure space, then show that*

a. $f_1^{\alpha_1} f_2^{\alpha_2} \cdots f_n^{\alpha_n} \in L_1(\mu)$, *and*
b. $\int f_1^{\alpha_1} f_2^{\alpha_2} \cdots f_n^{\alpha_n}\,d\mu \le (\|f_1\|_1)^{\alpha_1}(\|f_2\|_1)^{\alpha_2}\cdots(\|f_n\|_1)^{\alpha_n}$.

Solution. We shall establish the result by using induction on n. For $n = 1$ the result is trivial. For $n = 2$, note that $f_1^{\alpha_1} \in L_{\frac{1}{\alpha_1}}(\mu)$ and $f_2^{\alpha_2} \in L_{\frac{1}{\alpha_2}}(\mu)$. Since $\big(\frac{1}{\alpha_1}\big)^{-1} + \big(\frac{1}{\alpha_2}\big)^{-1} = \alpha_1 + \alpha_2 = 1$, it follows from Hölder's inequality that $f_1^{\alpha_1} f_2^{\alpha_2} \in L_1(\mu)$ and that

$$\begin{aligned}
\int f_1^{\alpha_1} f_2^{\alpha_2}\,d\mu &\le \Big(\int (f_1^{\alpha_1})^{\frac{1}{\alpha_1}}\,d\mu\Big)^{\alpha_1}\Big(\int (f_2^{\alpha_2})^{\frac{1}{\alpha_2}}\,d\mu\Big)^{\alpha_2}\\
&= \big(\|f_1\|_1\big)^{\alpha_1}\big(\|f_2\|_1\big)^{\alpha_2}.
\end{aligned}$$

For the inductive argument, assume that the result is true for some n. Let $f_1, \ldots, f_n, f_{n+1}$ be $n+1$ integrable positive functions and let $\alpha_1, \ldots, \alpha_n, \alpha_{n+1}$ be positive constants such that $\sum_{i=1}^{n+1} \alpha_i = 1$. Put $\alpha = \sum_{i=1}^n \alpha_i > 0$, and note that $\sum_{i=1}^n \frac{\alpha_i}{\alpha} = 1$. Now, by our induction hypothesis, we have $f_1^{\frac{\alpha_1}{\alpha}} \cdots f_n^{\frac{\alpha_n}{\alpha}} \in L_1(\mu)$. Also, applying the case $n = 2$ for α and $1 - \alpha = \alpha_{n+1}$, we see that

$$\begin{aligned}
\int f_1^{\alpha_1} \cdots f_n^{\alpha_n} f_{n+1}^{\alpha_{n+1}}\,d\mu &= \int \Big(f_1^{\frac{\alpha_1}{\alpha}} \cdots f_n^{\frac{\alpha_n}{\alpha}}\Big)^{\alpha} f_n^{\alpha_{n+1}}\,d\mu\\
&\le \Big(\int f_1^{\frac{\alpha_1}{\alpha}} \cdots f_n^{\frac{\alpha_n}{\alpha}}\,d\mu\Big)^{\alpha}\Big(\int f_{n+1}\,d\mu\Big)^{\alpha_{n+1}}\\
&\le \Big[(\|f_1\|_1)^{\frac{\alpha_1}{\alpha}} \cdots (\|f_n\|_1)^{\frac{\alpha_n}{\alpha}}\Big]^{\alpha}\big(\|f_{n+1}\|_1\big)^{\alpha_{n+1}}\\
&= \big(\|f_1\|_1\big)^{\alpha_1} \cdots \big(\|f_n\|_1\big)^{\alpha_n}\big(\|f_{n+1}\|_1\big)^{\alpha_{n+1}},
\end{aligned}$$

and the induction is complete.

Problem 31.27. *Let $(X, \mathcal{S}, \mu)$ be a measure space and let $\{A_n\}$ be a sequence of measurable sets satisfying $0 < \mu^*(A_n) < \infty$ for each n and $\lim \mu^*(A_n) = 0$. Fix $1 < p < \infty$ and let $g_n = [\mu^*(A_n)]^{-\frac{1}{q}} \chi_{A_n}$ $(n = 1, 2, \ldots)$, where $\frac{1}{p} + \frac{1}{q} = 1$. Prove that $\lim \int f g_n \, d\mu = 0$ for each $f \in L_p(\mu)$.*

Solution. Pick $1 < q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and let $f \in L_p(\mu)$. Then, by Hölder's inequality, we have

$$\begin{aligned}\left|\int f g_n \, d\mu\right| &= \left|\int (f\chi_{A_n}) g_n \, d\mu\right| \\ &\le \left(\int |f\chi_{A_n}|^p \, d\mu\right)^{\frac{1}{p}} \left(\int |g_n|^q \, d\mu\right)^{\frac{1}{q}} = \left(\int_{A_n} |f|^p \, d\mu\right)^{\frac{1}{p}}.\end{aligned}$$

From Problem 22.6, we know that $\lim \int_{A_n} |f|^p \, d\mu = 0$, and therefore $\lim \int f g_n \, d\mu = 0$ likewise holds.

Problem 31.28. *Let $(X, \mathcal{S}, \mu)$ be a measure space such that $\mu^*(X) = 1$. For each $1 < p < \infty$ define the set*

$$\mathcal{E}_p = \left\{ f \in L_1(\mu) \colon \int |f| \, d\mu = 1 \ \ \textit{and} \ \ \int |f|^p \, d\mu = 2 \right\}.$$

Show that for each $0 < \epsilon < 1$ there exists some $\delta_p > 0$ such that

$$\mu^*(\{x \in X \colon |f(x)| > \epsilon\}) \ge \delta_p$$

for each $f \in \mathcal{E}_p$.

Solution. Fix $0 < \varepsilon < 1$. For each $f \in \mathcal{E}_p$ put

$$E_f = \{x \in X \colon |f(x)| > \varepsilon\} \text{ and } F_f = X \setminus E_f = \{x \in X \colon |f(x)| \le \varepsilon\}.$$

From $|f|\chi_{F_f} \le \varepsilon \chi_{F_f}$, it follows that $\int_{F_f} |f| \, d\mu \le \varepsilon \mu^*(F_f) \le \varepsilon$, and so

$$\int_{E_f} |f| \, d\mu = \int_X |f| \, d\mu - \int_{F_f} |f| \, d\mu \ge 1 - \varepsilon. \qquad (\star)$$

Now, if $1 < q < \infty$ satisfies $\frac{1}{p} + \frac{1}{q} = 1$, then Hölder's inequality implies

$$\int_{E_f} |f|\,d\mu \le \Big(\int_{E_f} |f|^p\,d\mu\Big)^{\frac{1}{p}} \cdot \Big(\int_{E_f} 1^q\,d\mu\Big)^{\frac{1}{q}}$$
$$= \Big(\int_{E_f} |f|^p\,d\mu\Big)^{\frac{1}{p}} [\mu^*(E_f)]^{\frac{1}{q}} \le 2^{\frac{1}{p}} [\mu^*(E_f)]^{\frac{1}{q}}.$$

A glance at $(\star)$ shows that $1 - \varepsilon \le 2^{\frac{1}{p}} [\mu^*(E_f)]^{\frac{1}{q}}$, or

$$\mu^*(E_f) \ge \tfrac{(1-\varepsilon)^q}{2^{\frac{q}{p}}} = \tfrac{(1-\varepsilon)^q}{2^{q-1}} = 2\big(\tfrac{1-\varepsilon}{2}\big)^q$$

holds for each $f \in \mathcal{E}_p$, and the desired conclusion follows.

Problem 31.29. *Let $(X, \mathcal{S}, \mu)$ be a measure space and let $1 \le p < \infty$ and $0 < \eta < p$.*

a. *Show that the nonlinear function $\psi: L_p(\mu) \to L_{\frac{p}{\eta}}(\mu)$, where $\psi(f) = |f|^\eta$, is norm continuous.*
b. *If $f_n \to f$ and $g_n \to g$ hold in $L_p(\mu)$, then show that*

$$\lim_{n\to\infty} \int |f_n|^{p-\eta} |g_n|^\eta\,d\mu = \int |f|^{p-\eta} |g|^\eta\,d\mu.$$

Solution. (a) It should be clear that ψ maps indeed $L_p(\mu)$ into $L_{\frac{p}{\eta}}(\mu)$, and that ψ is nonlinear. Let $f_n \to f$ in $L_p(\mu)$ (i.e., let $\|f_n - f\|_p \to 0$) and assume by way of contradiction that $\psi(f_n) \not\to \psi(f)$ in $L_{\frac{p}{\eta}}(\mu)$. So, by passing to a subsequence, we can assume that there exists some $\varepsilon > 0$ such that

$$\|\psi(f_n) - \psi(f)\|_{\frac{p}{\eta}} = \Big(\int \big||f_n|^\eta - |f|^\eta\big|^{\frac{p}{\eta}}\,d\mu\Big)^{\frac{\eta}{p}} \ge \varepsilon. \qquad (\star\star)$$

Now, by passing to a subsequence again, we can assume that there exists some function $0 \le g \in L_p(\mu)$ such that $|f_n| \le g$ μ-a.e. holds for each n and $f_n \longrightarrow f$ a.e.; see Lemma 31.6. Therefore, the relations $\big||f_n|^\eta - |f|^\eta\big|^{\frac{p}{\eta}} \le \big(|g|^\eta + |f|^\eta\big)^{\frac{p}{\eta}} \in L_1(\mu)$ and $\big||f_n|^\eta - |f|^\eta\big|^{\frac{p}{\eta}} \longrightarrow 0$ a.e., coupled with the Lebesgue Dominated Convergence Theorem, imply $\int \big||f_n|^\eta - |f|^\eta\big|^{\frac{p}{\eta}}\,d\mu \longrightarrow 0$, which contradicts $(\star\star)$. Consequently, the nonlinear mapping ψ is norm continuous.
(b) Notice that the two nonlinear functions $\psi_1: L_p(\mu) \to L_{\frac{p}{p-\eta}}(\mu)$ and $\psi_2: L_p(\mu) \to L_{\frac{p}{\eta}}(\mu)$, defined by

$$\psi_1(u) = |u|^{p-\eta} \quad\text{and}\quad \psi_2(v) = |v|^\eta.$$

are—by part (a)—both norm continuous. Therefore,

$$\big\| |f_n|^{p-\eta} - |f|^{p-\eta} \big\|_{\frac{p}{p-\eta}} \to 0 \quad \text{and} \quad \big\| |g_n|^{\eta} - |g|^{\eta} \big\|_{\frac{p}{\eta}} \to 0.$$

Now, observe that $\big(L_{\frac{p}{p-\eta}}(\mu)\big)^* = L_{\frac{p}{\eta}}(\mu)$ holds. Consequently, from the duality $\langle L_{\frac{p}{p-\eta}}(\mu), L_{\frac{p}{\eta}}(\mu)\rangle$, we see that

$$\int |f_n|^{p-\eta}|g_n|^{\eta}\, d\mu = \langle |f_n|^{p-\eta}, |g_n|^{\eta}\rangle \to \langle |f|^{p-\eta}, |g|^{\eta}\rangle = \int |f|^{p-\eta}|g|^{\eta}\, d\mu,$$

as claimed.

Problem 31.30. *Let $T\colon L_p(\mu) \to L_p(\mu)$ be a continuous operator, where $1 < p < \infty$, and let $0 \le \eta \le p$. Show that:*

a. *If $f \in L_p(\mu)$, then $|f|^{p-\eta}|Tf|^{\eta} \in L_1(\mu)$ and*

$$\int |f|^{p-\eta}|Tf|^{\eta}\, d\mu \le \|T\|^{\eta}(\|f\|_p)^p.$$

b. *If for some $f \in L_p(\mu)$ with $\|f\|_p \le 1$ we have $\int |f|^{p-\eta}|Tf|^{\eta}\, d\mu = \|T\|^{\eta}$, then $|Tf| = \|T\||f|$.*

Solution. Assume T, η, and f are as stated in the problem.

(a) If $\eta = 0$ or $\eta = p$, then the desired inequality is obvious. So, assume $0 < \eta < p$ and consider the conjugate exponents

$$r = \tfrac{p}{p-\eta} \quad \text{and} \quad s = \big(1 - \tfrac{1}{r}\big)^{-1} = \tfrac{p}{\eta}.$$

Since $|f|^{p-\eta} \in L_r(\mu)$ and $|Tf|^{\eta} \in L_s(\mu)$, we see that $|f|^{p-\eta}|Tf|^{\eta}$ belongs to $L_1(\mu)$. Also, applying Hölder's inequality with exponents r and s, we obtain

$$\begin{aligned}
\int |f|^{p-\eta}|Tf|^{\eta}\, d\mu &\le \Big[\int \big(|f|^{p-\eta}\big)^{\frac{p}{p-\eta}} d\mu\Big]^{\frac{p-\eta}{p}} \cdot \Big[\int \big(|Tf|^{\eta}\big)^{\frac{p}{\eta}} d\mu\Big]^{\frac{\eta}{p}} \\
&= \Big(\int |f|^p\, d\mu\Big)^{\frac{p-\eta}{p}} \cdot \Big(\int |Tf|^p\, d\mu\Big)^{\frac{\eta}{p}} \\
&= (\|f\|_p)^{p-\eta}(\|Tf\|_p)^{\eta} \le (\|f\|_p)^{p-\eta}\|T\|^{\eta}(\|f\|_p)^{\eta} \\
&= \|T\|^{\eta}(\|f\|_p)^p.
\end{aligned}$$

(b) Assume that some $f \in L_p(\mu)$ with $\|f\|_p \le 1$ satisfies

$$\int |f|^{p-\eta}|Tf|^\eta \, d\mu = \|T\|^\eta.$$

From Hölder's inequality, we see that

$$\begin{aligned}\|T\|^\eta = \int |f|^{p-\eta}|Tf|^\eta \, d\mu &\le \left\||f|^{p-\eta}\right\|_r \cdot \left\||Tf|^\eta\right\|_s \\ &= (\|f\|_p)^{p-\eta}(\|Tf\|_p)^\eta \le \|T\|^\eta.\end{aligned}$$

Thus, $\int |f|^{p-\eta}|Tf|^\eta \, d\mu = \left\||f|^{p-\eta}\right\|_r \cdot \left\||Tf|^\eta\right\|_s$. From Problem 31.4, there exists a constant $c \ge 0$ such that $\left(|Tf|^\eta\right)^s = c\left(|f|^{p-\eta}\right)^r$, or

$$|Tf|^p = c|f|^p.$$

Therefore, $|Tf| = \lambda|f|$ holds for some $\lambda \ge 0$. This implies $\lambda\|f\|_p = \|Tf\|_p \le \|T\|\|f\|_p$ and so $\lambda \le \|T\|$. Also, from

$$\|T\|^\eta = \int |f|^{p-\eta}|Tf|^\eta \, d\mu = \int |f|^{p-\eta}\lambda^\eta|f|^\eta \, d\mu \le \lambda^\eta,$$

we see that $\|T\| \le \lambda$. Hence, $\lambda = \|T\|$, and so $|Tf| = \|T\||f|$.

Problem 31.31. *Let (X, S, μ) be a measure space and let $f \in L_p(\mu)$ for some $1 \le p < \infty$. Show that the function $g\colon [0, \infty) \to [0, \infty]$ defined by*

$$g(t) = pt^{p-1}\mu^*\left(\{x \in X\colon |f(x)| \ge t\}\right)$$

is Lebesgue integrable over $[0, \infty)$ and that

$$\int |f|^p \, d\mu = \int_{[0,\infty)} g(t)\, d\lambda(t) = p\int_0^\infty t^{p-1}\mu^*(\{x \in X\colon |f(x)| \ge t\})\, dt.$$

Solution. Let $f \in L_p(\mu)$; we shall assume that $f(x) \in \mathbf{R}$ holds for each $x \in X$. Let $T = [0, \infty)$ and consider the product measure space $T \times X$. Also, let

$$A = \{(t, x) \in T \times X\colon 0 \le t \le |f(x)|\},$$

and note that (by Problem 26.8) the set A is $\mu \times \lambda$-measurable. Now, consider

the function $h: T \times X \longrightarrow [0, \infty)$ defined by

$$h(t,x) = \begin{cases} pt^{p-1} & \text{if } 0 \le t \le |f(x)| \\ 0 & \text{if } t > |f(x)| \end{cases} = pt^{p-1}\chi_A(t,x).$$

In addition, we have

$$\begin{aligned}\int_X \Big[\int_T h(t,x)\,d\lambda(t)\Big]\,d\mu(x) &= \int_X \Big[\int_T pt^{p-1}\chi_A(t,x)\,d\lambda(t)\Big]\,d\mu(x) \\ &= \int_X \Big[\int_0^{|f(x)|} pt^{p-1}\,dt\Big]\,d\mu(x) \\ &= \int_X |f(x)|^p\,d\mu(x) < \infty.\end{aligned}$$

By Tonneli's Theorem (Theorem 26.7), the function h is integrable over $T \times X$ and

$$\int_X |f(x)|^p\,d\mu(x) = \int_T \Big[\int_X h(t,x)\,d\mu(x)\Big]\,d\lambda(t). \qquad (\star)$$

Put $E_t = \{x \in X\colon\ |f(x)| \ge t\}$ and note that

$$\int_X h(t,x)\,d\mu(x) = \int_{E_t} pt^{p-1}\,d\mu(x) = pt^{p-1}\mu^*(E_t).$$

By Fubini's Theorem (Theorem 26.6), we know that the function

$$t \longmapsto pt^{p-1}\mu^*(E_t)$$

is integrable over $[0, \infty)$ and from $(\star)$, we see that

$$\int_X |f(x)|^p\,d\mu(x) = p\int_T t^{p-1}\mu^*(E_t)\,d\lambda(t) = p\int_0^\infty t^{p-1}\mu^*(E_t)\,dt.$$

That the Lebesgue integral $\int_T t^{p-1}\mu^*(E_t)\,d\lambda(t)$ is also an improper Riemann integral follows from the fact that the function $t \longmapsto \mu^*(E_t)$ is decreasing—and hence continuous for all but at-most countably many t.

Problem 31.32. *Let $(X, \mathcal{S}, \mu)$ be a measure space and let $f: X \to \mathbf{R}$ be a measurable function. If $\mu^*\big(\{x \in X\colon\ |f(x)| \ge t\}\big) \le e^{-t}$ for all $t \ge 0$, then show that $f \in L_p(\mu)$ holds for each $1 \le p < \infty$.*

Solution. Let $g(t) = \mu^*(\{x \in X\colon |f(x)| \geq t\})$, $t \geq 0$. In view of Problem 31.31, we must show that $\int_0^\infty t^{p-1}g(t)\,d\lambda(t) < \infty$. Since for each $t \geq 0$ we have $0 \leq g(t) \leq e^{-t}$, it suffices to establish that $\int_0^\infty t^{p-1}e^{-t}\,dt < \infty$.

To see this, start by observing that by L'Hôpital's Rule we have

$$\lim_{t\to\infty} t^{p-1}e^{-\frac{t}{2}} = \lim_{t\to\infty} \frac{t^{p-1}}{e^{\frac{t}{2}}} = 0.$$

So, there exists some $M > 0$ satisfying $0 \leq t^{p-1}e^{-\frac{t}{2}} \leq M$ for all $t \geq 0$. Hence,

$$\begin{aligned} 0 \leq \int_0^\infty t^{p-1}e^{-t}\,dt &= \int_0^\infty t^{p-1}e^{-\frac{t}{2}}e^{-\frac{t}{2}}\,dt \\ &\leq \int_0^\infty Me^{-\frac{t}{2}}\,dt = 2M < \infty, \end{aligned}$$

as desired.

Problem 31.33. *Consider the vector space of functions*

$$E = \Big\{f\colon \mathbf{R}^n \to \mathbf{R} \,|\, f \text{ is a } C^\infty\text{-function with compact support and } \int_{\mathbf{R}^n} f\,d\lambda = 0\Big\}.$$

Show that for each $1 < p < \infty$ the vector space E is dense in $L_p(\mathbf{R}^n)$. Is E dense in $L_1(\mathbf{R}^n)$?

Solution. We shall prove the result for the special case $n = 1$. The general case (whose details can be completed as in Problem 25.3) is left for the reader. The proof will be based upon the following property: If $1 < p < \infty$, $\varepsilon > 0$, $h > 0$, and a positive integer n are given, then there exists a C^∞-function $\phi\colon \mathbf{R} \longrightarrow \mathbf{R}$ such that

1. $\operatorname{Supp}\phi$ is compact and $\operatorname{Supp}\phi \subseteq [n, \infty)$;
2. $0 \leq \phi(x) \leq h$ for all $x \in \mathbf{R}$;
3. $\int_{\mathbf{R}}\phi\,d\lambda = 1$; and
4. $\|\phi\|_p = \big(\int_{\mathbf{R}}\phi^p\,d\lambda\big)^{\frac{1}{p}} < \varepsilon$.

To see this, assume $1 < p < \infty$, $\varepsilon > 0$, $h > 0$, and the positive integer n are given. If k is an arbitrary positive integer, then (by Problem 25.3) there exists a C^∞-function $f\colon \mathbf{R} \longrightarrow \mathbf{R}$ such that:

a. $\operatorname{Supp} f \subseteq [n, n+k+2]$;

b. $0 \le f(x) \le 1$ for each $x \in \mathbf{R}$ and $f(x) = 1$ for each $x \in [n+1, n+k+1]$.

If $c = \int_{\mathbf{R}} f\,d\lambda > 0$, then the C^∞-function $\phi = \frac{1}{c}f$ satisfies $\operatorname{Supp}\phi \subseteq [n, n+k+2] \subseteq [n, \infty)$, $\int_{\mathbf{R}}\phi\,d\lambda = 1$, and (in view of $c = \int_{\mathbf{R}} f\,d\lambda \ge \int_{n+1}^{n+k+1} 1\,dx = k$) $0 \le \phi(x) \le \frac{1}{k}$ for each $x \in \mathbf{R}$. In addition, we have

$$\|\phi\|_p = \Big(\int_{\mathbf{R}} \phi^p\,d\lambda\Big)^{\frac{1}{p}} \le \Big[\int_n^{n+k+2} (\tfrac{1}{k})^p\,d\lambda\Big]^{\frac{1}{p}}$$
$$= \frac{(k+2)^{\frac{1}{p}}}{k} = \big(\tfrac{k+2}{k}\big)^{\frac{1}{p}} \cdot k^{\frac{1}{p}-1}.$$

In view of $1 < p < \infty$, we see that $\lim_{k\to\infty}\big(\frac{k+2}{k}\big)^{\frac{1}{p}} \cdot k^{\frac{1}{p}-1} = 0$, and so a sufficiently large k will yield a function ϕ with the desired properties.

To complete the proof, let $f \in L^p(\mathbf{R})$ and let $\varepsilon > 0$. As in Problem 25.5(b) (how?), there exists a C^∞-function g with compact support such that $\|f-g\|_p < \varepsilon$. If $m = \int_{\mathbf{R}} g\,d\lambda = 0$, then $g \in E$, and we are done. So, assume that $m \ne 0$. Pick a positive integer n such that $\operatorname{Supp} g \cap [n, \infty) = \emptyset$, and then (by the prior discussion) pick a C^∞-function ϕ with compact support such that:

i. $\operatorname{Supp}\phi \subseteq [n, \infty)$;
ii. $\phi(x) \ge 0$ for each $x \in \mathbf{R}$;
iii. $\int_{\mathbf{R}}\phi\,d\lambda = 1$; and
iv. $\|\phi\|_p = \big(\int_{\mathbf{R}}\phi^p\,d\lambda\big)^{\frac{1}{p}} < \frac{\varepsilon}{|m|}$.

Now, consider the function $\psi = g - m\phi$, and note that $\psi \in E$ and

$$\begin{aligned}\|f-\psi\|_p &= \|(f-g)+m\phi\|_p \\ &\le \|f-g\|_p + |m|\|\phi\|_p \\ &< \varepsilon + |m|\|\phi\|_p < \varepsilon + \varepsilon = 2\varepsilon.\end{aligned}$$

Therefore, E is dense in $L_p(\mathbf{R})$.

The vector space E is not dense in $L_1(\mathbf{R})$. For instance, consider the function $f = \chi_{[0,1]} \in L_1(\mathbf{R})$. If $\phi \in L_1(\mathbf{R})$ satisfies $\int |f-\phi|\,d\lambda < \frac{1}{2}$, then from

$$1 - \int_{\mathbf{R}} \phi\,d\lambda = \int_{\mathbf{R}} (f-\phi)\,d\lambda \le \int_{\mathbf{R}} |f-\phi|\,d\lambda < \tfrac{1}{2},$$

it follows that $\int_{\mathbf{R}}\phi\,d\lambda > 1 - \frac{1}{2} = \frac{1}{2}$, and so $\phi \notin E$. This shows that E is not dense in $L_1(\mathbf{R})$.

Problem 31.34. *Let $(0,\infty)$ be equipped with the Lebesgue measure, and let $1 < p < \infty$. For each $f \in L_p(\lambda)$ let*

$$T(f)(x) = x^{-1}\int f\chi_{(0,x)}\,d\lambda \quad \textit{for } x > 0.$$

Then show that T defines a one-to-one bounded linear operator from $L_p((0,\infty))$ into itself such that $\|T\| = \frac{p}{p-1}$.

Solution. For simplicity, we shall write Tf instead of $T(f)$. Consider an arbitrary function $0 \le f \in C_c\big((0,\infty)\big)$. Choose some $M > 0$ so that $0 \le f(x) \le M$ holds for all $x > 0$.

If $I = \int_0^\infty f(t)\,dt$, then the function

$$g(x) = \begin{cases} M & \text{if } 0 < x \le 1 \\ \frac{I}{x} & \text{if } x > 1 \end{cases}$$

belongs to $L_p\big((0,\infty)\big)$. Since $0 \le Tf \le g$ holds, we see that Tf belongs to $L_p\big((0,\infty)\big)$. Also, in view of the inequalities

$$0 \le x\Big(\tfrac{1}{x}\int_0^x f(t)\,dt\Big)^p = x[Tf(x)]^p \le x[g(x)]^p,$$

it follows that

$$x\Big(\tfrac{1}{x}\int_0^x f(t)\,dt\Big)^p\Big|_0^\infty = 0.$$

Now, integrating by parts and using Hölder's inequality, we get

$$\begin{aligned}
\big(\|Tf\|_p\big)^p &= \int_0^\infty\Big(\tfrac{1}{x}\int_0^x f(t)\,dt\Big)^p dx = \tfrac{1}{1-p}\int_0^\infty\Big(\int_0^x f(t)\,dt\Big)^p d(x^{1-p}) \\
&= \tfrac{1}{1-p}\Big[x\Big(\tfrac{1}{x}\int_0^x f(t)dt\Big)^p\Big|_0^\infty - p\int_0^\infty f(x)\Big(\tfrac{1}{x}\int_0^x f(t)dt\Big)^{p-1}dx\Big] \\
&= \tfrac{p}{p-1}\int_0^\infty f(x)[Tf(x)]^{p-1}\,dx \\
&\le \tfrac{p}{p-1}\|f\|_p\Big(\int_0^\infty [Tf(x)]^{q(p-1)}\,dx\Big)^{\frac{1}{q}} \\
&= \tfrac{p}{p-1}\|f\|_p\cdot\big(\|Tf\|_p\big)^{\frac{p}{q}}.
\end{aligned}$$

This easily implies that

$$\|Tf\|_p \leq \tfrac{p}{p-1}\|f\|_p$$

holds for all $f \in C_c\big((0,\infty)\big)$. In other words,

$$T\colon C_c\big((0,\infty)\big) \longrightarrow L_p\big((0,\infty)\big)$$

defines a continuous operator such that $\|T\| \leq \frac{p}{p-1}$ holds.

Since $C_c\big((0,\infty)\big)$ is norm dense in $L_p\big((0,\infty)\big)$ (Theorem 31.11), T has a unique continuous (linear) extension T^* to all of $L_p\big((0,\infty)\big)$ such that $\|T^*\| \leq \frac{p}{p-1}$ holds. Our next objective is to show that $T^*f(x) = \frac{1}{x}\int_0^x f(t)\,d\lambda(t) = Tf(x)$ holds for all $f \in L_p\big((0,\infty)\big)$ and all $x > 0$.

To this end, let $0 \leq \phi$ be a step function. Choose some $C > 0$ satisfying $0 \leq \phi(x) \leq C$ for all $x > 0$. By Theorem 31.11, there exists a sequence $\{f_n\}$ of $C_c\big((0,\infty)\big)$ with $\lim \int |f_n - \phi|^p\,d\lambda = 0$. We can assume that $\lim f_n(x) = \phi(x)$ holds for almost all x (see Lemma 31.6). In view of

$$\big|f_n \wedge C - \phi\big| = \big|f_n \wedge C - \phi \wedge C\big| \leq \big|f_n - C\big|,$$

replacing $\{f_n\}$ by $\{f_n \wedge C\}$, we can assume that $0 \leq f_n(x) \leq C$ holds for all $x > 0$ and all n. Since $\lim \|Tf_n - T^*\phi\|_p = 0$, we can also assume (by passing to a subsequence) that $Tf_n(x) \longrightarrow T^*\phi(x)$ holds for almost all x. Next, observe that for each fixed $x > 0$ we have $\phi \in L_1\big((0,x)\big)$ and so, by the Lebesgue Dominated Convergence Theorem, we see that

$$T^*\phi(x) = \lim_{n\to\infty} Tf_n(x) = \lim_{n\to\infty} \frac{1}{x}\int_0^x f_n(t)\,d\lambda(t) = \frac{1}{x}\int_0^x \phi(t)\,d\lambda(t)$$

holds for almost all x. Now, let $0 \leq f \in L_p\big((0,\infty)\big)$. Choose a sequence $\{\phi_n\}$ of step functions with $0 \leq \phi_n \uparrow f$. In view of

$$\lim \|T^*\phi_n - T^*f\|_p = 0,$$

we can assume that $T^*\phi_n(x) \longrightarrow T^*f(x)$ holds for almost all x. Taking into account that for each fixed $x > 0$, we have $f \in L_p\big((0,x)\big) \subseteq L_1\big((0,x)\big)$, the Lebesgue Dominated Convergence Theorem implies

$$T^*f(x) = \lim_{n\to\infty} T^*\phi_n(x) = \lim_{n\to\infty} \frac{1}{x}\int_0^x \phi_n(t)\,d\lambda(t) = \frac{1}{x}\int_0^x f(t)\,d\lambda(t)$$

holds for almost all x. Thus, $T^* = T$ holds.

Next, we shall show that $\|T\| = \frac{p}{p-1}$ holds. We already know that $\|T\| \leq \frac{p}{p-1}$ holds. So, it must be established that $\|T\| \geq \frac{p}{p-1}$. To this end, let

$$f_n(x) = \begin{cases} x^{(n^{-1}-1)p^{-1}} & \text{if } 0 < x < 1 \\ 0 & \text{if } x \geq 1 \,. \end{cases}$$

Then, $\big(\|f_n\|_p\big)^p = \int_0^1 x^{n^{-1}-1}\,dx = n$, and moreover,

$$Tf_n(x) = \frac{np}{1+n(p-1)} \begin{cases} x^{(n^{-1}-1)p^{-1}} & \text{if } 0 < x < 1 \\ x^{-1} & \text{if } x \geq 1 \,. \end{cases}$$

Consequently, we have

$$\begin{aligned} \big(\|Tf_n\|_p\big)^p &= \int_0^\infty |Tf_n(x)|^p \, d\lambda(x) \\ &= \Big[\tfrac{np}{1+n(p-1)}\Big]^p \Big[\int_0^1 x^{n^{-1}-1}\,dx + \int_1^\infty x^{-p}\,dx\Big] \\ &= \Big[\tfrac{np}{1+n(p-1)}\Big]^p \big(n + \tfrac{1}{p-1}\big), \end{aligned}$$

and so

$$\tfrac{np}{1+n(p-1)}\big[n + \tfrac{1}{p-1}\big]^{\frac{1}{p}} = \|Tf_n\|_p \leq \|T\| \cdot \|f_n\|_p = \|T\| \cdot n^{\frac{1}{p}}.$$

This implies

$$\|T\| \geq \tfrac{p}{p-1+\frac{1}{n}} \cdot \big[1 + \tfrac{1}{n(p-1)}\big]^{\frac{1}{p}} \longrightarrow \tfrac{p}{p-1},$$

from which it follows that $\|T\| \geq \frac{p}{p-1}$ also holds.

Finally, we establish that T is one-to-one. Assume that $Tf = 0$ holds for some $f \in L_p\big((0,\infty)\big)$. Then, $\int_0^x f(t)\,d\lambda(t) = 0$ holds for all $x > 0$. Now, by Problem 22.19, we infer that $f = 0$ a.e. holds, and so the operator T is one-to-one.

CHAPTER 6

HILBERT SPACES

32. INNER PRODUCT SPACES

Problem 32.1. *Let $c_1, c_2, \ldots, c_n$ be n (strictly) positive real numbers. Show that the function of two variables $(\cdot,\cdot)\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, defined by $(x,y) = \sum_{i=1}^n c_i x_i y_i$, is an inner product on $\mathbb{R}^n$.*

Solution. Notice that for all vectors $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$ and $z = (z_1, \ldots, z_n)$ in $\mathbb{R}^n$ we have

$$(\alpha x + \beta y, z) = \sum_{i=1}^n c_i(\alpha x_i + \beta y_i)z_i = \alpha \sum_{i=1}^n c_i x_i z_i + \beta \sum_{i=1}^n c_i y_i z_i = \alpha(x,z) + \beta(y,z),$$

$$(x,y) = \sum_{i=1}^n c_i x_i y_i = \sum_{i=1}^n c_i y_i x_i = (y,x), \quad \text{and}$$

$$(x,x) = \sum_{i=1}^n c_i x_i^2 \geq 0.$$

Moreover, $(x,x) = \sum_{i=1}^n c_i x_i^2 = 0$ implies $c_i x_i^2 = 0$ for each i, and so (since $c_i > 0$ for each i) $x_i = 0$ for each i, i.e., $x = 0$. The above show the function $(\cdot,\cdot)$ is an inner product on $\mathbb{R}^n$.

Problem 32.2. *Let $(X, (\cdot,\cdot))$ be a real inner product vector space with complexification $X_{\mathbf{c}}$. Show that the function $\langle\cdot,\cdot\rangle\colon X_{\mathbf{c}} \times X_{\mathbf{c}} \to \mathbf{C}$ defined via the formula*

$$\langle x + \imath y, x_1 + \imath y_1\rangle = (x, x_1) + (y, y_1) + \imath[(y, x_1) - (x, y_1)].$$

is an inner product on $X_{\mathbf{c}}$. Also, show that the norm induced by the inner product $\langle\cdot,\cdot\rangle$ on $X_{\mathbf{c}}$ is given by

$$\|x + \imath y\| = \sqrt{(x,x) + (y,y)} = (\|x\|^2 + \|y\|^2)^{\frac{1}{2}}.$$

Solution. Let $x_1 + \imath y_1, x_2 + \imath y_2, x_3 + \imath y_3 \in X_c$. We check below the properties of the inner product.

1. (Additivity)

$$\begin{aligned}
&\langle (x_1 + \imath y_1) + (x_2 + \imath y_2), x_3 + \imath y_3 \rangle \\
&= \langle x_1 + x_2 + \imath (y_1 + y_2), x_3 + \imath y_3 \rangle \\
&= (x_1 + x_2, x_3) + (y_1 + y_2, y_3) + \imath \big[(y_1 + y_2, x_3) - (x_1 + x_2, y_3) \big] \\
&= (x_1, x_3) + (y_1, y_3) + \imath [(y_1, x_3) - (x_1, y_3)] + ((x_2, x_3) + (y_2, y_3) \\
&\qquad + \imath [(y_2, x_3) - (x_2, y_3)]) \\
&= \langle x_1 + \imath y_1, x_3 + \imath y_3 \rangle + \langle x_2 + \imath y_2, x_3 + \imath y_3 \rangle .
\end{aligned}$$

2. (Homogeneity)

$$\begin{aligned}
&\langle (\alpha + \imath \beta)(x_1 + \imath y_1), x_2 + \imath y_2 \rangle \\
&= \langle \alpha x_1 - \beta y_1 + \imath (\beta x_1 + \alpha y_1), x_2 + \imath y_2 \rangle \\
&= (\alpha x_1 - \beta y_1, x_2) + (\beta x_1 + \alpha y_1, y_2) + \imath [(\beta x_1 + \alpha y_1, x_2) \\
&\qquad - (\alpha x_1 - \beta y_1, y_2)] \\
&= (\alpha + \imath \beta) \big[(x_1, x_2) + (y_1, y_2) + \imath [(y_1, x_2) - (x_1, y_2)] \big] \\
&= (\alpha + \imath \beta) \langle x_1 + \imath y_1, x_2 + \imath y_2 \rangle .
\end{aligned}$$

3. (Conjugate Linearity)

$$\begin{aligned}
\overline{\langle x_1 + \imath y_1, x_2 + \imath y_2 \rangle} &= \overline{(x_1, x_2) + (y_1, y_2) + \imath [(y_1, x_2) - (x_1, y_2)]} \\
&= (x_1, x_2) + (y_1, y_2) + \imath [(x_1, y_2) - (y_1, x_2)] \\
&= (x_2, x_1) + (y_2, y_1) + \imath [(y_2, x_1) - (x_2, y_1)] \\
&= \langle x_2 + \imath y_2, x_1 + \imath y_1 \rangle .
\end{aligned}$$

4. (Positivity)

$$\langle x_1 + \imath y_1, x_1 + \imath y_1 \rangle = (x_1, x_1) + (y_1, y_1) \geq 0.$$

Moreover,

$$\langle x_1 + \imath y_1, x_1 + \imath y_1 \rangle = (x_1, x_1) + (y_1, y_1) = 0 \iff x_1 = y_1 = 0 \iff x_1 + \imath y_1 = 0.$$

Problem 32.3. *Let Ω be a Hausdorff compact topological space and let μ be a regular Borel measure on Ω such that* $\operatorname{Supp}\mu = \Omega$. *Show that the function $(\cdot,\cdot)\colon C(\Omega)\times C(\Omega)\to \mathbb{R}$, defined by*

$$(f,g) = \int_\Omega fg\,d\mu,$$

is an inner product. Also, describe the complexification of $C(\Omega)$ and the extension of the inner product to the complexification of $C(\Omega)$.

Solution. If $f, g, h \in C(\Omega)$ and $\alpha, \beta \in \mathbb{R}$, then note that

$$\begin{aligned}(\alpha f+\beta g, h) &= \int_\Omega(\alpha f+\beta g)h\,d\mu = \alpha\int_\Omega fh\,d\mu+\beta\int_\Omega gh\,d\mu = \alpha(f,h)+\beta(g,h),\\ (f,g) &= \int_\Omega fg\,d\mu = \int_\Omega gf\,d\mu = (g,f), \text{ and}\\ (f,f) &= \int_\Omega f^2\,d\mu \geq 0.\end{aligned}$$

Moreover, observe that (since $\operatorname{Supp}\mu = \Omega$) a function $f \in C(\Omega)$ satisfies

$$(f,f) = \int_\Omega f^2\,d\mu \iff f = 0.$$

The complexification $C_c(\Omega)$ of $C(\Omega)$ consists of all complex-valued functions $f + \imath g$, where $f, g \in C(\Omega)$. The complex inner product is given by

$$\langle f, g\rangle = \int_\Omega f\overline{g}\,d\mu$$

for all $f, g \in C_c(\Omega)$.

Problem 32.4. *Show that equality holds in the Cauchy–Schwarz inequality (i.e., $|(x,y)| = \|x\|\,\|y\|$) if and only if x and y are linearly dependent vectors.*

Solution. Assume $|(x,y)| = \|x\|\,\|y\|$. If $x = 0$, then the conclusion is obvious. So, assume $x \neq 0$. Let $(x,y) = re^{\imath\theta}$. Replacing x by $e^{-\imath\theta}x$, we can assume without loss of generality that $(x,y) = r \geq 0$, and so $(x,y) = \|x\|\,\|y\|$. Now,

notice that for each real λ we have

$$\begin{aligned} 0 \le (\lambda x + y, \lambda x + y) &= \lambda^2 \|x\|^2 + \lambda\big[(x, y) + \overline{(x, y)}\big] + \|y\|^2 \\ &= \lambda^2 \|x\|^2 + 2\lambda(x, y) + \|y\|^2 \\ &= \lambda^2 \|x\|^2 + 2\lambda \|x\|\,\|y\| + \|y\|^2 \\ &= \big(\lambda\|x\| + \|y\|\big)^2. \end{aligned}$$

So, if $\lambda = -\frac{\|y\|}{\|x\|}$, then $(\lambda x + y, \lambda x + y) = 0$ or $\lambda x + y = 0$. This implies $\|y\|x - \|x\|y = 0$, which means that the vectors x and y are linearly dependent.

If x and y are linearly dependent, then the equation $|(x, y)| = \|x\|\,\|y\|$ should be obvious.

Problem 32.5. *If x is a vector in an inner product space, then show that*

$$\|x\| = \sup_{\|y\|=1} |(x, y)|.$$

Solution. If $x = 0$, then the conclusion is obvious. So, we consider the case $x \neq 0$. If $\|y\| = 1$, then the Cauchy–Schwarz inequality implies $|(x, y)| \le \|x\|\,\|y\| \le \|x\|$. Therefore, we have

$$\sup_{\|y\|=1} |(x, y)| \le \|x\|.$$

For the reverse inequality, let $z = x/\|x\|$. Then, $\|z\| = 1$, and so

$$\sup_{\|y\|=1} |(x, y)| \ge |(x, z)| = |(x, x/\|x\|)| = (x, x)/\|x\| = \|x\|.$$

Therefore, $\|x\| = \sup_{\|y\|=1} |(x, y)|$, with the supremum being in actuality the maximum.

Problem 32.6. *Show that in a real inner product space $x \perp y$ holds if and only if $\|x + y\|^2 = \|x\|^2 + \|y\|^2$. Does $\|x + y\|^2 = \|x\|^2 + \|y\|^2$ in a complex inner product space imply $x \perp y$?*

Solution. Let x and y be two vectors in a real inner product space. If $x \perp y$, then the Pythagorean Theorem gives $\|x + y\|^2 = \|x\|^2 + \|y\|^2$. Conversely, if

$\|x+y\|^2 = \|x\|^2 + \|y\|^2$, then from

$$\begin{aligned}\|x+y\|^2 &= (x+y, x+y) = (x,x) + (x,y) + (y,x) + (y,y)\\ &= \|x\|^2 + (x,y) + (y,x) + \|y\|^2\\ &= \|x\|^2 + 2(x,y) + \|y\|^2,\end{aligned}$$

it follows that $2(x, y) = 0$, and so $x \perp y$.

In complex inner product space the Pythagorean identity $\|x+y\|^2 = \|x\|^2+\|y\|^2$ does not imply $x \perp y$. To see this, consider a non-zero vector x and let $y = \imath x$. Clearly, $\|y\|^2 = (\imath x, \imath x) = \|x\|^2$. Now, note that

$$\begin{aligned}\|x+y\|^2 &= \|x+\imath x\|^2 = \|(1+\imath)x\|^2 = |1+\imath|^2\,\|x\|^2\\ &= 2\|x\|^2 = \|x\|^2 + \|y\|^2,\end{aligned}$$

while $(x, y) = (x, \imath x) = -\imath\|x\|^2 \neq 0$.

Problem 32.7. *Assume that a sequence $\{x_n\}$ in an inner product space satisfies $(x_n, x) \to \|x\|^2$ and $\|x_n\| \to \|x\|$. Show that $x_n \to x$.*

Solution. Observe that $(x_n, x) \to \|x\|^2$ implies $(x, x_n) = \overline{(x_n, x)} \to \overline{\|x\|^2} = \|x\|^2$. So, from

$$\begin{aligned}\|x_n - x\|^2 &= (x_n - x, x_n - x)\\ &= \|x_n\|^2 - (x, x_n) - (x_n, x) + \|x\|^2\\ &\longrightarrow \|x\|^2 - \|x\|^2 - \|x\|^2 + \|x\|^2 = 0,\end{aligned}$$

it follows that $\|x_n - x\| \to 0$, i.e., $x_n \to x$.

Problem 32.8. *Let S be an orthogonal subset of an inner product space. Show that there exists a complete orthogonal subset C such that $S \subseteq C$.*

Solution. Assume that S is an orthogonal subset of an inner product space X. Let $\mathcal{C}$ denote the collection of all orthogonal sets that contain S. That is, an orthogonal set A of vectors of X belongs to $\mathcal{C}$ if and only if $S \subseteq A$. If we consider $\mathcal{C}$ partially ordered by the inclusion relation $\subseteq$, then it is easy to see that $\mathcal{C}$ satisfies the hypotheses of Zorn's Lemma. Now, notice that any maximal element C of $\mathcal{C}$ is a complete orthogonal set satisfying $S \subseteq C$.

Problem 32.9. *Show that the norms of the following Banach spaces cannot be induced by inner products.*

a. *The norm* $\|x\| = \max\{|x_1|, |x_2|, \ldots, |x_n|\}$ *on* $\mathbf{R}^n$.
b. *The sup norm on* $C[a, b]$.
c. *The* L_p*-norm on any* $L_p(\mu)$*-space for each* $1 \le p \le \infty$ *with* $p \ne 2$.

Solution. (a) Consider the vectors $x = (1, 0, \ldots, 0)$ and $y = (0, 1, 0, \ldots, 0)$. Clearly,

$$\|x\| = \|y\| = \|x + y\| = \|x - y\| = 1,$$

and so

$$\|x + y\|^2 + \|x - y\|^2 = 2 \quad \text{and} \quad 2\|x\|^2 + 2\|y\|^2 = 4.$$

Therefore, $\|x + y\|^2 + \|x - y\|^2 \ne 2\|x\|^2 + 2\|y\|^2$ and consequently the norm $\|\cdot\|$ does not satisfy the Parallelogram Law. This implies that the norm $\|\cdot\|$ cannot be induced by an inner product.

(b) Again, we shall show that the sup norm $\|\cdot\|_\infty$ does not satisfy the Parallelogram Law—and this will guarantee that the sup norm is not induced by an inner product. To see this, consider the two functions $\mathbf{1}$ (the constant function one) and $f: [a, b] \to \mathbf{R}$ defined by $f(x) = \frac{x-a}{b-a}$. Now, note that

$$\|\mathbf{1}\|_\infty = \|f\|_\infty = 1, \quad \|\mathbf{1} + f\|_\infty = 2, \quad \text{and} \quad \|\mathbf{1} - f\|_\infty = 1.$$

Therefore,

$$(\|\mathbf{1} + f\|_\infty)^2 + (\|\mathbf{1} - f\|_\infty)^2 = 5 \ne 4 = 2(\|\mathbf{1}\|_\infty)^2 + 2(\|f\|_\infty)^2,$$

so that the norm $\|\cdot\|_\infty$ does not satisfy the Parallelogram Law.

(c) Assume that there are two disjoint measurable sets E and F such that $0 < \mu^*(E) < \infty$ and $0 < \mu^*(F) < \infty$. First, we consider the case $p = \infty$. Then, note that

$$\|\chi_E\|_\infty = \|\chi_F\|_\infty = 1 \quad \text{and} \quad \|\chi_E + \chi_F\|_\infty = \|\chi_E - \chi_F\|_\infty = 1,$$

and consequently,

$$(\|\chi_E + \chi_F\|_\infty)^2 + (\|\chi_E - \chi_F\|_\infty)^2 = 2 \ne 4 = 2(\|\chi_E\|_\infty)^2 + 2(\|\chi_F\|_\infty)^2.$$

This shows that the norm $\|\cdot\|_\infty$ does not satisfy the Parallelogram Law and so is not induced by an inner product.

Now, consider the case $1 \le p < \infty$ with $p \neq 2$. The functions $f = [\mu^*(E)]^{-\frac{1}{p}}\chi_E$ and $g = [\mu^*(F)]^{-\frac{1}{p}}\chi_F$ satisfy $\|f\|_p = \|g\|_p = 1$, and hence,

$$2(\|f\|_p)^2 + 2(\|g\|_p)^2 = 4 = 2^2.$$

Also, from $|f+g|^p = |f-g|^p = |f|^p + |g|^p$, we see that

$$(\|f+g\|_p)^2 + (\|f-g\|_p)^2 = \left(2^{\frac{1}{p}}\right)^2 + \left(2^{\frac{1}{p}}\right)^2 = 2\left(2^{\frac{1}{p}}\right)^2 = 2^{1+\frac{2}{p}}$$

Since $p \neq 2$, we have $2^{1+\frac{2}{p}} \neq 2^2$, and so

$$(\|f+g\|_p)^2 + (\|f-g\|_p)^2 \neq 2(\|f\|_p)^2 + 2(\|g\|_p)^2.$$

This shows that the norm $\|\cdot\|_p$ does not satisfy the Parallelogram Law and so it is not induced by an inner product.

Problem 32.10. *Show that a norm $\|\cdot\|$ in a complex vector space is induced by an inner product if and only if it satisfies the Parallelogram Law, i.e., if and only if*

$$\|x+y\|^2 + \|x-y\|^2 = 2\big(\|x\|^2 + \|y\|^2\big)$$

holds for all vectors x and y. Moreover, show that if $\|\cdot\|$ satisfies the Parallelogram Law, then the inner product $(\cdot,\cdot)$ that induces $\|\cdot\|$ is given by

$$(x,y) = \frac{1}{4}\big(\|x+y\|^2 - \|x-y\|^2 + \imath\|x+\imath y\|^2 - \imath\|x-\imath y\|^2\big).$$

Solution. If $\|\cdot\|$ is induced by the inner product $(\cdot,\cdot)$, then for all vectors x and y, we have

$$\begin{aligned}\|x+y\|^2 + \|x-y\|^2 &= (x+y,x+y) + (x-y,x-y)\\ &= [(x,x)+(y,x)+(x,y)+(y,y)]\\ &\quad + [(x,x)-(y,x)-(x,y)+(y,y)]\\ &= 2\|x\|^2 + 2\|y\|^2.\end{aligned}$$

For the converse, assume that the norm $\|\cdot\|$ satisfies the Parallelogram Law. Consider, the complex-valued function $(\cdot,\cdot)$ defined by

$$(x,y) = \frac{1}{4}\big(\|x+y\|^2 - \|x-y\|^2 + \imath\|x+\imath y\|^2 - \imath\|x-\imath y\|^2\big).$$

Clearly, $(x,x) = \|x\|^2$ holds for all vectors x. To finish the solution, we shall

verify that $(\cdot, \cdot)$ is a complex inner product. Start by observing that

$$\begin{aligned}(y, x) &= \frac{1}{4}\big(\|y+x\|^2 - \|y-x\|^2 + \imath\|y+\imath x\|^2 - \imath\|y-\imath x\|^2\big)\\ &= \frac{1}{4}\big(\|x+y\|^2 - \|x-y\|^2 - \imath\|(-\imath)(x+\imath y)\|^2 + \imath\|\imath(x-\imath y)\|^2\big)\\ &= \frac{1}{4}\big(\|x+y\|^2 - \|x-y\|^2 - \imath\|x+\imath y\|^2 + \imath\|x-\imath y\|^2\big)\\ &= \overline{(x, y)}.\end{aligned}$$

Next, note that for all vectors u, v and w, we have

$$\begin{aligned}&4(u+v, w) + 4(u-v, w)\\ &= \big[\|u+v+w\|^2 - \|u+v-w\|^2 + \imath\|u+v+\imath w\|^2 - \imath\|u+v-\imath w\|^2\big]\\ &\quad + \big[\|u-v+w\|^2 - \|u-v-w\|^2 + \imath\|u-v+\imath w\|^2 - \imath\|u-v-\imath w\|^2\big]\\ &= \big[\|u+w+v\|^2 + \|u+w-v\|^2\big] - \big[\|u-w+v\|^2 + \|u-w-v\|^2\big]\\ &\quad + \imath\big[\|u+\imath w+v\|^2 + \|u+\imath w-v\|^2\big] - \imath\big[\|u-\imath w+v\|^2 + \|u-\imath w-v\|^2\big]\\ &= 2\|u+w\|^2 + 2\|v\|^2 - 2\|u-w\|^2 - 2\|v\|^2\\ &\quad + \imath\big[2\|u+iw\|^2 + 2\|v\|^2 - 2\|u-\imath w\|^2 - 2\|v\|^2\big]\\ &= 2\big[\|u+w\|^2 - \|u-w\|^2 + \imath\|u+\imath w\|^2 - \imath\|u-\imath w\|^2\big]\\ &= 8(u, w).\end{aligned}$$

Thus, for all vectors u, v, and w we have

$$(u+v, w) + (u-v, w) = 2(u, w). \qquad (\star)$$

When $v = u$, $(\star)$ yields $(2u, w) = 2(u, w)$. Now, letting $u = \frac{1}{2}(x+y)$, $v = \frac{1}{2}(x-y)$ and $w = z$ in $(\star)$, we get

$$(x, z) + (y, z) = (u+v, z) + (u-v, z) = 2(u, z) = (2u, z) = (x+y, z),$$

which is the additivity of $(\cdot, \cdot)$ in the first variable.

For the homogeneity, note first that

$$\begin{aligned}(\imath x, y) &= \frac{1}{4}\big(\|\imath x+y\|^2 - \|\imath x-y\|^2 + \imath\|\imath x+\imath y\|^2 - \imath\|\imath x-\imath y\|^2\big)\\ &= \frac{1}{4}\big(\|\imath x+y\|^2 - \|\imath x-y\|^2 + \imath\|x+y\|^2 - \imath\|x-y\|^2\big)\\ &= \imath\Big[\frac{1}{4}\big(\|x+y\|^2 - \|x-y\|^2 - \imath\|\imath x+y\|^2 + \imath\|\imath x-y\|^2\big)\Big]\end{aligned}$$

$$= \imath\left[\frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 - \imath\|\imath(x-\imath y)\|^2 + \imath\|\imath(x+\imath y)\|^2\right)\right]$$
$$= \imath\left[\frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 + \imath\|x+\imath y\|^2 - \imath\|x-\imath y\|^2\right)\right]$$
$$= \imath(x, y).$$

Now, as in the proof of Lemma 18.7, we can establish that $(rx, y) = r(x, y)$ holds for each "real" rational number r and all $x, y \in X$. Since $(\cdot, \cdot)$, as defined above, is a jointly continuous function (relative to the norm $\|\cdot\|$), it easily follows that $(\alpha x, y) = \alpha(x, y)$ holds for all $\alpha \in \mathbf{R}$ and all $x, y \in X$. Finally, for an arbitrary complex number $\alpha + \imath\beta$ and arbitrary vectors x and y, note that

$$\begin{aligned}\big((\alpha+\imath\beta)x, y\big) &= (\alpha x + \imath\beta x, y) = (\alpha x, y) + \big(\beta(\imath x), y\big)\\ &= \alpha(x, y) + \beta(\imath x, y) = \alpha(x, y) + \beta\imath(x, y)\\ &= (\alpha+\imath\beta)(x, y).\end{aligned}$$

This establishes that $(\cdot, \cdot)$ is an inner product that induces the norm $\|\cdot\|$.

Problem 32.11. *Let X be a complex inner product space and let $T: X \to X$ be a linear operator. Show that $T = 0$ if and only if $(Tx, x) = 0$ for each $x \in X$. Is this result true for real inner product spaces?*

Solution. Assume that $(Tx, x) = 0$ holds for all $x \in X$. From the identity

$$\big(T(x+y), x+y\big) = (Tx, x) + (Tx, y) + (Ty, x) + (Ty, y)$$

and our hypothesis, it follows that

$$(Tx, y) + (Ty, x) = 0 \qquad (\star\star)$$

for all $x, y \in X$. Replacing y by $\imath y$ in $(\star\star)$ yields $(Tx, \imath y) + (T(\imath y), x) = \imath\big[-(Tx, y) + (Ty, x)\big] = 0$. So,

$$-(Tx, y) + (Ty, x) = 0 \qquad (\star\star\star)$$

holds for all $x, y \in X$. Adding $(\star\star)$ and $(\star\star\star)$, we get $2(Ty, x) = 0$ or $(Ty, x) = 0$ for all $x, y \in X$. Letting $x = Ty$, we get $(Ty, Ty) = 0$ and so $Ty = 0$ for all $y \in X$, i.e. $T = 0$.

For real inner product spaces the preceding conclusion is false. Here is an example. Consider the Euclidean space $\mathbf{R}^2$ equipped with its standard inner product and define the linear operator $T: \mathbf{R}^2 \to \mathbf{R}^2$ by $T(x) = (-x_2, x_1)$ for all

$x = (x_1, x_2) \in \mathbb{R}^2$. Clearly, $T \neq 0$, and

$$(Tx, x) = (-x_2, x_1) \cdot (x_1, x_2) = -x_2x_1 + x_1x_2 = 0$$

holds for all $x \in \mathbb{R}^2$.

Problem 32.12. *If $\{x_n\}$ is an orthonormal sequence in an inner product space, then show that* $\lim(x_n, y) = 0$ *for each vector* y.

Solution. Let $\{x_n\}$ be an orthonormal sequence in an inner product space, and let y be an arbitrary vector. Then, from Bessel's Inequality, we have

$$\sum_{n=1}^{\infty} |(x_n, y)|^2 \leq \|y\|^2 < \infty.$$

This implies $|(x_n, y)|^2 \to 0$, and so $(x_n, y) \to 0$.

Problem 32.13. *The* **orthogonal complement** *of a nonempty subset A of an inner product space X is defined by*

$$A^\perp = \{x \in X\colon\ x \perp y \text{ for all } y \in A\}.$$

We shall denote $(A^\perp)^\perp$ by $A^{\perp\perp}$. Establish the following properties regarding orthogonal complements:

a. *$A^\perp$ is a closed subspace of X, $A \subseteq A^{\perp\perp}$ and $A \cap A^\perp = \{0\}$.*
b. *If $A \subseteq B$, then $B^\perp \subseteq A^\perp$.*
c. *$A^\perp = \overline{A}^\perp = [\mathcal{L}(A)]^\perp = [\overline{\mathcal{L}(A)}]^\perp$, where $\mathcal{L}(A)$ denotes the vector subspace generated by A in X.*
d. *If M and N are two vector subspaces of X, then $M^{\perp\perp} + N^{\perp\perp} \subseteq (M+N)^{\perp\perp}$.*
e. *If M is a finite dimensional subspace, then $X = M \oplus M^\perp$.*

Solution. (a) If $x, y \in A^\perp$ and α, β are arbitrary scalars, then for each $z \in A$ we have

$$(\alpha x + \beta y, z) = \alpha(x, z) + \beta(y, z) = \alpha\, 0 + \beta\, 0 = 0,$$

and so $\alpha x + \beta y \in A^\perp$. Therefore, $A^\perp$ is a vector subspace of X. Since $x \in A$ implies $x \perp y$ for all $y \in A^\perp$, it follows that $x \in A^{\perp\perp}$, i.e., $A \subseteq A^{\perp\perp}$. Now if $x \in A \cap A^\perp$, then $(x, x) = 0$ or $x = 0$, and thus $A \cap A^\perp = \{0\}$.

(b) Assume $A \subseteq B$ and $x \in B^\perp$. If $y \in A$, then $y \in B$, and so $y \perp x$. This implies $x \in A^\perp$, and so $B^\perp \subseteq A^\perp$.

(c) From $A \subseteq \overline{A}$ and Part (b), it follows that $\overline{A}^{\perp} \subseteq A^{\perp}$. Now, let $x \in A^{\perp}$ and let $y \in \overline{A}$. Pick a sequence $\{y_n\} \subseteq A$ satisfying $y_n \to y$ and note that

$$(y, x) = \lim_{n\to\infty} (y_n, x) = 0.$$

Therefore, $x \in \overline{A}^{\perp}$. Hence, $A^{\perp} \subseteq \overline{A}^{\perp}$, and thus $A^{\perp} = \overline{A}^{\perp}$.

For the other equalities, note first that $A \subseteq \mathcal{L}(A)$ implies $[\mathcal{L}(A)]^{\perp} \subseteq A^{\perp}$. Now, fix $x \in A^{\perp}$, and let $y \in \mathcal{L}(A)$. Pick $y_1, \dots, y_k \in A$ and scalars $\lambda_1, \dots, \lambda_k$ such that $y = \sum_{i=1}^{k} \lambda_i y_i$. Then,

$$(y, x) = \Big(\sum_{i=1}^{k} \lambda_i y_i, x\Big) = \sum_{i=1}^{k} \lambda_i (y_i, x) = 0.$$

This shows that $x \in [\mathcal{L}(A)]^{\perp}$. Thus, $A^{\perp} \subseteq [\mathcal{L}(A)]^{\perp}$, and so $A^{\perp} = [\mathcal{L}(A)]^{\perp}$.

(d) From $M \subseteq M + N$, it follows that $M^{\perp\perp} \subseteq (M+N)^{\perp\perp}$ holds. Likewise, $N \subseteq M+N$ implies $N^{\perp\perp} \subseteq (M+N)^{\perp\perp}$. Therefore, $M^{\perp\perp} + N^{\perp\perp} \subseteq (M+N)^{\perp\perp}$.

(e) Let M be a finite dimensional subspace of dimension n. In order to establish that $X = M \oplus M^{\perp}$, we must show that every vector can be written in the form $y + z$ with $y \in M$ and $y \in M^{\perp}$. (The uniqueness of the decomposition should be obvious.)

Start by fixing a Hamel basis $\{x_1, x_2, \dots, x_n\}$ of M. Replacing (if necessary) $\{x_1, x_2, \dots, x_n\}$ by the normalized set of vectors that can be obtained by applying the Gram–Schmidt orthogonalization process (Theorem 32.11) to $\{x_1, x_2, \dots, x_n\}$, we can assume that the set $\{x_1, x_2, \dots, x_n\}$ is also an orthonormal set.

Now, fix $x \in X$ and consider the vectors

$$z = \sum_{k=1}^{n} (x, x_k)x_k \quad \text{and} \quad y = x - \sum_{k=1}^{n} (x, x_k)x_k.$$

Clearly, $z \in M$ and since $(y, x_k) = 0$ for each k, it easily follows that $y \in M^{\perp}$. Now, note that $x = y + z \in M \oplus M^{\perp}$.

Problem 32.14. *Let V be a vector subspace of a real inner product space X. A linear operator $L: V \to X$ is said to be* **symmetric** *if $(Lx, y) = (x, Ly)$ holds for all $x, y \in V$.*

a. *Consider the real inner product space $C[a, b]$ and let $V = \{f \in C^2[a, b]: f(a) = f(b) = 0\}$. Also, let $p \in C^1[a, b]$ and $q \in C[a, b]$ be two fixed functions. Show that the linear operator $L: V \to C[a, b]$, defined by*

$$L(f) = (pf')' + qf,$$

is a symmetric operator.

b. *Consider $\mathbb{R}^n$ equipped with its standard inner product and let $A: \mathbb{R}^n \to \mathbb{R}^n$ be a linear operator. As usual, we identify the operator with the matrix $A = [a_{ij}]$ representing it, where the jth column of the matrix A is the column vector Ae_j. Show that A is a symmetric operator if and only if A is a symmetric matrix. (Recall that an $n \times n$ matrix $B = [b_{ij}]$ is said to be* **symmetric** *if $b_{ij} = b_{ji}$ holds for all i and j.)*

c. *Let $L: V \to X$ be a symmetric operator. Then L extends naturally to a linear operator $L: V_c = \{x + iy: x, y \in V\} \to X_c$ via the formula $L(x + iy) = Lx + iLy$. Show that L also satisfies $(Lu, v) = (u, Lv)$ for all $u, v \in V_c$ and that the eigenvalues of L are all real numbers.*

d. *Show that eigenvectors of a symmetric operator corresponding to distinct eigenvalues are orthogonal.*

Solution. (a) If $f, g \in V$, then note that

$$\begin{aligned}
(Lf, g) &= \int_a^b \big([p(x)f'(x)]' + q(x)f(x)\big)g(x)\,dx \\
&= \int_a^b [p(x)f'(x)]'g(x)\,dx + \int_a^b q(x)f(x)g(x)\,dx \\
&= p(x)f'(x)g(x)\Big|_a^b - \int_a^b p(x)f'(x)g'(x)\,dx + \int_a^b q(x)f(x)g(x)\,dx \\
&= \int_a^b q(x)f(x)g(x)\,dx - \int_a^b p(x)f'(x)g'(x)\,dx \\
&= (f, Lg).
\end{aligned}$$

(b) Recall that the transpose of a matrix $B = [b_{ij}]$ is the matrix $B^t = [b_{ji}]$. In terms of the transpose, a matrix A is symmetric if and only if $A^t = A$. Now, our conclusion follows immediately from the following two identities:

$$(Ax, y) = (x, A^t y) \text{ for all } x, y \in \mathbb{R}^n, \quad \text{and}$$
$$a_{ij} = (e_i, Ae_j).$$

(c) If $u = x + iy$ and $v = x_1 + iy_1$ are vectors of V_c, then note that

$$\begin{aligned}
(Lu, v) &= \big(L(x + iy), x_1 + iy_1\big) = \big(Lx + iLy, x_1 + iy_1\big) \\
&= (Lx, x_1) + (Ly, y_1) + i\big[(Ly, x_1) - (Lx, y_1)\big] \\
&= (x, Lx_1) + (y, Ly_1) + i\big[(y, Lx_1) - (x, Ly_1)\big] \\
&= (x + iy, Lx_1 + iLy_1) = (u, Lv).
\end{aligned}$$

Now, assume that $\lambda \in \mathbf{C}$ is an eigenvalue of $L: V_c \to X_c$. Fix a unit vector $u \in X_c$ satisfying $Lu = \lambda u$, and note that

$$\lambda = \lambda(u, u) = (\lambda u, u) = (Lu, u) = (u, Lu) = (u, \lambda u) = \overline{\lambda}(u, u) = \overline{\lambda}.$$

This shows that λ is a real number.

(d) Assume that $L: V \to X$ is a symmetric operator and let two nonzero vectors $u, v \in V_c$ satisfy $Lu = \lambda u$ and $Lv = \mu v$ with $\lambda \neq \mu$. By part (c), we know that λ and μ are real numbers. Therefore,

$$(\lambda - \mu)(u, v) = \lambda(u, v) - \mu(u, v) = (\lambda u, v) - (u, \mu v) = (Lu, v) - (u, Lv) = 0,$$

and so $(u, v) = 0$.

Problem 32.15. *Let $(\cdot, \cdot)$ denote the standard inner product on $\mathbf{R}^n$, i.e., $(x, y) = \sum_{i=1}^{n} x_i y_i$ for all $x, y \in \mathbf{R}^n$. Recall that an $n \times n$ matrix A is said to be* **positive definite** *if $(x, Ax) > 0$ holds for all nonzero vectors $x \in \mathbf{R}^n$.*

Show that a function of two variables $\langle \cdot, \cdot \rangle: \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ is an inner product on $\mathbf{R}^n$ if and only if there exists a unique real symmetric positive definite matrix A such that

$$\langle x, y \rangle = (x, Ay)$$

holds for all $x, y \in \mathbf{R}^n$. (It is known that a symmetric matrix is positive definite if and only if its eigenvalues are all positive.)

Solution. Let $\langle \cdot, \cdot \rangle: \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ be a function of two variables. Assume first that there exists a real symmetric positive definite matrix A such that

$$\langle x, y \rangle = (x, Ay)$$

holds for all $x, y, z \in \mathbf{R}^n$. Then, for all $x, y \in \mathbf{R}^n$ and all $\alpha, \beta \in \mathbf{R}$, we have

$$\begin{aligned}
\langle x, y \rangle &= (x, Ay) = (Ax, y) = (y, Ax) = \langle y, x \rangle, \\
\langle \alpha x + \beta y, z \rangle &= (\alpha x + \beta y, Az) = \alpha(x, Az) + \beta(y, Az) = \alpha\langle x, z \rangle + \beta\langle y, z \rangle, \text{ and} \\
\langle x, x \rangle &= (x, Ax) \geq 0 \text{ for all } x \in \mathbf{R}^n \text{ and } \langle x, x \rangle = 0 \iff x = 0.
\end{aligned}$$

This shows that $\langle \cdot, \cdot \rangle$ is an inner product.

For the converse assume that the function of two variables $\langle \cdot, \cdot \rangle: \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ is a real inner product. Let $e_1, e_2, \ldots, e_n$ denote the standard unit vectors, and so

each vector $x \in \mathbf{R}^n$ is written as $x = \sum_{i=1}^n x_i e_i$. It follows that

$$\langle x, y\rangle = \Big\langle \sum_{i=1}^n x_i e_i, \sum_{j=1}^n y_j e_j \Big\rangle = \sum_{i=1}^n \sum_{j=1}^n x_i y_j \langle e_i, e_j\rangle = (x, Ay)$$

for all $x, y \in \mathbf{R}^n$, where A is the $n \times n$ matrix $A = \big[\langle e_i, e_j\rangle\big]$. Clearly, A is a real symmetric matrix and in view of $(x, Ax) = \langle x, x\rangle$, we see that A is also a positive definite matrix. The uniqueness of A should be obvious.

33. HILBERT SPACES

Problem 33.1. *Let $(X, \mathcal{S}, \mu)$ be a measure space and let $\rho: X \to (0, \infty)$ be a measurable function—called a* **weight function**. *Show that the collection of measurable functions*

$$L_2(\rho) = \Big\{ f \in \mathcal{M}: \int \rho |f|^2\, d\mu < \infty \Big\}$$

under the inner product $(\cdot, \cdot): L_2(\rho) \times L_2(\rho) \to \mathbf{R}$, defined by

$$(f, g) = \int \rho f g\, d\mu,$$

is a real Hilbert space.

Solution. It should be clear that $L_2(\rho)$ is a vector space. Moreover, since $f \in L_2(\rho)$ is equivalent to $\sqrt{\rho} f \in L_2(\mu)$, it follows from Hölder's inequality that

$$\Big| \int \rho f g\, d\mu \Big| \le \Big(\int \rho |f|^2\, d\mu \Big)^{\frac{1}{2}} \Big(\int \rho |g|^2\, d\mu \Big)^{\frac{1}{2}} < \infty,$$

and so $(\cdot, \cdot)$ is well-defined. We leave it as an exercise for the reader to verify that $(\cdot, \cdot)$ is indeed a real inner product. We shall prove that $L_2(\rho)$ is a Hilbert space by establishing that it is complete.

To this end, let $\{f_n\} \subseteq L_2(\rho)$ be a Cauchy sequence. That is, for each $\epsilon > 0$ there exists some n_0 such that

$$\|f_n - f_m\|^2 = \int \rho |f_n - f_m|^2\, d\mu = \int \big|\sqrt{\rho} f_n - \sqrt{\rho} f_m\big|^2\, d\mu < \epsilon^2$$

holds for all $n, m \geq n_0$. This means that the sequence of functions $\{\sqrt{\rho} f_n\} \subseteq L_2(\mu)$ is a norm Cauchy sequence of the Hilbert space $L_2(\mu)$. Since $L_2(\mu)$ is a Banach space (Theorem 31.5), it follows that there exists some function $g \in L_2(\mu)$ such that

$$\int |\sqrt{\rho} f_n - g|^2 \, d\mu \longrightarrow 0.$$

Now, note that if $f = g/\sqrt{\rho}$, then $f \in L_\rho(\mu)$ and

$$\|f_n - f\|^2 = \int \rho |f_n - f|^2 \, d\mu = \int |\sqrt{\rho} f_n - g|^2 \, d\mu \longrightarrow 0.$$

This shows that $L_2(\rho)$ is norm complete and hence, it is a Hilbert space.

The reader should also notice that $L_2(\rho)$ is exactly the Hilbert space $L_2(\nu)$ for the measure $\nu\colon \Lambda_\mu \to [0, \infty]$ defined by

$$\nu(A) = \int_A \rho(x) \, d\mu(x)$$

for each $A \in \Lambda_\mu$.

Problem 33.2. *Show that the Hilbert space $L_2[0, \infty)$ is separable.*

Solution. Consider the countable set of functions $\{f_{k,n}\colon k, n = 1, 2, \ldots\}$, where

$$f_{k,n}(x) = \begin{cases} x^n & \text{if } 0 \leq x \leq k \\ 0 & \text{if } k < x\,. \end{cases}$$

We know that the continuous functions with compact support are dense in $L^2[0, \infty)$ (Theorem 31.11) and so, we need only prove that the linear span of $\{f_{k,n}\}$ is dense in the vector space of continuous functions with compact support. Observe that if this is established, then the linear span of $\{f_{k,n}\}$ with rational coefficients would be a countable dense set.

Let $f \in L_2[0, \infty)$ be a continuous function with compact support, and let $\epsilon > 0$. Fix an integer k such that $f(x) = 0$ for all $x \geq k$. By the Stone-Weierstrass approximation theorem, there exists a polynomial $P(x) = \sum_{n=0}^m c_n x^n$ satisfying $|f(x) - P(x)| < \epsilon/\sqrt{k}$ for each $x \in [0, k]$. Now, notice that if we consider the function $g \in L_2[0, \infty)$ defined by $g(x) = \sum_{n=0}^m c_n f_{k,n}(x)$, then

$$\begin{aligned} \|f - g\|_2 &= \Big(\int_0^\infty |f(x) - g(x)|^2 \, dx\Big)^{\frac{1}{2}} = \Big(\int_0^k |f(x) - P(x)|^2 \, dx\Big)^{\frac{1}{2}} \\ &< \Big(\int_0^k \tfrac{\epsilon^2}{k} \, dx\Big)^{\frac{1}{2}} = \epsilon \end{aligned}$$

holds. This shows that the linear span of the countable set $\{f_{k,n}\colon k, n = 1, 2, \ldots\}$ is dense in $L_2[0, \infty)$, and so $L_2[0, \infty)$ is separable.

Problem 33.3. *Let $\{\psi_n\}$ be an orthonormal sequence of functions in the Hilbert space $L_2[a, b]$ which is also uniformly bounded. If $\{\alpha_n\}$ is a sequence of scalars such that $\alpha_n \psi_n \to 0$ a.e., then show that* $\lim \alpha_n = 0$.

Solution. Fix some constant C such that $|\psi_n(x)| \le C$ hold for all n and for all $x \in [a, b]$. Also, let $\{\alpha_n\}$ be a sequence of scalars such that $\alpha_n \psi_n(x) \to 0$ holds for almost all x.

Next, fix $\epsilon > 0$ so that $\epsilon C^2 < \frac{1}{2}$. Now, by Egorov's Theorem 16.7, there exists a measurable set $E \subseteq [a, b]$ with $\lambda(E^c) < \epsilon$ such that the sequence of functions $\{\alpha_n \psi_n\}$ converges uniformly to zero on E. So, there exists an integer m such that $|\alpha_n \psi_n(x)| \le \epsilon$ for all $n \ge m$ and all $x \in E$. Then, we have

$$\begin{aligned}
|\alpha_n|^2 &= \int_a^b |\alpha_n \psi_n(t)|^2\,dt = \int_E |\alpha_n \psi_n(t)|^2\,dt + \int_{E^c} |\alpha_n \psi_n(t)|^2\,dt \\
&\le \int_E \epsilon^2\,dt + |\alpha_n|^2 \int_{E^c} |\psi_n(t)|^2\,dt \\
&\le \epsilon^2(b-a) + |\alpha_n|^2 \int_{E^c} C^2\,dt \\
&\le \epsilon^2(b-a) + \epsilon|\alpha_n|^2 C^2.
\end{aligned}$$

This implies $\frac{1}{2}|\alpha_n|^2 < (1 - \epsilon C^2)|\alpha_n|^2 \le \epsilon^2(b-a)$ for all $n \ge m$, or

$$|\alpha_n| \le \epsilon\sqrt{2(b-a)}$$

for all $n \ge m$. Since $0 < \epsilon < \frac{1}{2C^2}$ is arbitrary, we have established that $\alpha_n \to 0$.

Problem 33.4. *Let $\{\phi_n\}$ be an orthonormal sequence of functions in the Hilbert space $L_2[-1, 1]$. Show that the sequence of functions $\{\psi_n\}$, where*

$$\psi_n(x) = \left(\tfrac{2}{b-a}\right)^{\frac{1}{2}} \phi_n\left(\tfrac{2}{b-a}\left(x - \tfrac{b+a}{2}\right)\right),$$

is an orthonormal sequence in the Hilbert space $L_2[a, b]$.

Solution. Observe that the inner product satisfies

$$\begin{aligned}
(\psi_n, \psi_m) &= \int_a^b \psi_n(x)\,\overline{\psi_m}(x)\,dx \\
&= \tfrac{2}{b-a} \int_a^b \phi_n\left(\tfrac{2}{b-a}\left(x - \tfrac{b+a}{2}\right)\right) \overline{\phi_m}\left(\tfrac{2}{b-a}\left(x - \tfrac{b+a}{2}\right)\right) dx.
\end{aligned}$$

Making the substitution $t = \frac{2}{b-a}\left(x - \frac{b+a}{2}\right)$, we have $dt = \frac{2}{b-a}dx$, and so

$$(\psi_n, \psi_m) = \int_{-1}^{1} \phi_n(t)\overline{\phi_m}(t)\,dt = \delta_{mn}.$$

That is, $\{\psi_n\}$ is an orthonormal sequence in the Hilbert space $L_2[a, b]$.

Problem 33.5. *Show that the norm completion $\hat{X}$ of an inner product space X is a Hilbert space. Moreover, if $x, y \in \hat{X}$ and two sequences $\{x_n\}$ and $\{y_n\}$ of X satisfy $x_n \to x$ and $y_n \to y$ in $\hat{X}$, then establish that the inner product of $\hat{X}$ is given by*

$$\langle x, y\rangle = \lim_{n\to\infty} (x_n, y_n).$$

Solution. Assume that $\hat{X}$ is the norm completion of an inner product space X and let $x, y \in \hat{X}$. Pick two sequences $\{x_n\}$ and $\{y_n\}$ of X such that $\|x_n - x\| \to 0$ and $\|y_n - y\| \to 0$, where $\|\cdot\|$ is the norm of $\hat{X}$ (which is the unique continuous extension of the norm of X to $\hat{X}$). Fix some constant $M > 0$ such that $\|x_n\| \le M$ and $\|y_n\| \le M$ hold for each n. Then, using the Cauchy–Schwarz inequality, we have

$$\begin{aligned}
\big|(x_n, y_n) - (x_m, y_m)\big| &= \big|(x_n, y_n) - (x_n, y_m) + (x_n, y_m) - (x_m, y_m)\big| \\
&= \big|(x_n, y_n - y_m) + (x_n - x_m, y_m)\big| \\
&\le \big|(x_n, y_n - y_m)\big| + \big|(x_n - x_m, y_m)\big| \\
&\le \|x_n\|\,\|y_n - y_m\| + \|x_n - x_m\|\,\|y_m\| \\
&\le M\big(\|x_n - x_m\| + \|y_n - y_m\|\big).
\end{aligned}$$

This shows that the sequence of scalars $\{(x_n, y_n)\}$ is a Cauchy sequence and hence, convergent.

Next, assume two other sequences $\{x'_n\}$ and $\{y'_n\}$ of X satisfy $\|x'_n - x\| \to 0$ and $\|y'_n - y\| \to 0$. We can assume without loss of generality that $\|x'_n\| \le M$ and $\|y'_n\| \le M$ holds for each n. By the preceding $\lim(x'_n, y'_n)$ exists, and since

$$\begin{aligned}
\big|(x_n, y_n) - (x'_n, y'_n)\big| &= \big|(x_n, y_n) - (x_n, y'_n) + (x_n, y'_n) - (x'_n, y'_n)\big| \\
&= \big|(x_n, y_n - y'_n) + (x_n - x'_n, y'_n)\big| \\
&\le \big|(x_n, y_n - y'_n)\big| + \big|(x_n - x'_n, y'_n)\big| \\
&\le \|x_n\|\,\|y_n - y'_n\| + \|x_n - x'_n\|\,\|y'_n\| \\
&\le M\big(\|x_n - x'_n\| + \|y_n - y'_n\|\big) \longrightarrow 0,
\end{aligned}$$

it follows that $\lim(x_n, y_n) = \lim(x'_n, y'_n)$. In other words, the formula

$$\langle x, y\rangle = \lim_{n\to\infty} \langle x_n, y_n\rangle \qquad (\star)$$

gives rise to well-defined scalar-valued function on $\hat{X} \times \hat{X}$.

Now, it should be clear that the properties of the inner product are transferred via $(\star)$ from the inner product of X to the function $\langle\cdot,\cdot\rangle\colon \hat{X} \times \hat{X} \to \mathbf{C}$. In other words, the formula of $(\star)$ is an inner product on $\hat{X}$. Moreover, from

$$\|x\|^2 = \lim_{n\to\infty} \|x_n\|^2 = \lim_{n\to\infty} \langle x_n, x_n\rangle = \langle x, x\rangle,$$

we see that the inner product given by $(\star)$ induces the norm of $\hat{X}$.

Problem 33.6. *Show that the closed unit ball of ℓ_2 is not a norm compact set.*

Solution. Let $\mathcal{U} = \{x \in \ell_2\colon \|x\| \le 1\}$ be the closed unit ball of ℓ_2. Now, for each n let $e_n = (0, 0, \ldots, 0, 1, 0, 0, \ldots)$, the sequence with 1 in its nth coordinate and zero elsewhere. Note that $\|e_n\| = 1$ for each n and thus $\{e_n\}$ is a sequence of the unit ball of ℓ^2. (In fact $\{e_n\}$ is an orthonormal sequence of ℓ_2.) Now, notice that for $n \ne m$ we have $\|e_n - e_m\| = \sqrt{2}$. This implies that $\{e_n\}$ does not have any Cauchy subsequences—and hence, it does not have any convergent subsequences either. Now a glance at Theorem 7.3 guarantees that $\mathcal{U}$ is not a norm compact subset of ℓ_2.

Problem 33.7. *Show that the* **Hilbert cube** *(the set of all $x = (x_1, x_2, \ldots) \in \ell_2$ such that $|x_n| \le \frac{1}{n}$ holds for all n) is a compact subset of ℓ_2.*

Solution. Let $C = \big\{(x_1, x_2, \ldots) \in \ell_2\colon |x_n| \le \frac{1}{n} \text{ for each } n = 1, 2, \ldots\big\}$. Clearly, C is a closed subset of ℓ_2. Thus, in order to establish the compactness of C, it suffices to prove (by Theorem 7.8) that C is totally bounded.

To this end, let $\varepsilon > 0$. Fix some n such that $\sum_{k=n+1}^{\infty} \frac{1}{k^2} < \varepsilon$. Since the set $A = \big\{(x_1, \ldots, x_n) \in \mathbf{R}^n\colon |x_i| \le \frac{1}{i} \text{ for } 1 \le i \le n\big\}$ is closed and bounded, it must be a compact subset of $\mathbf{R}^n$. Pick $x^1, \ldots, x^m \in A$ $\big(\text{where } x^i = (x_1^i, \ldots, x_n^i)\big)$ so that $A \subseteq \bigcup_{i=1}^m B(x^i, \varepsilon)$ holds. (We consider, of course, $\mathbf{R}^n$ equipped with the Euclidean distance.) Now, for each $1 \le i \le m$ let $y_i = (x_1^i, \ldots, x_n^i, 0, 0, \ldots)$. Then, it is easy to see that $C \subseteq \bigcup_{i=1}^m B(y_i, 2\varepsilon)$ holds in ℓ_2. This shows that C is totally bounded, as required.

Problem 33.8. *Show that every subspace M of a Hilbert space satisfies $\overline{M} = M^{\perp\perp}$.*

Solution. It should be clear that $(\overline{M})^{\perp} = M^{\perp}$; see Problem 32.13. Therefore, by Theorem 33.7, $H = \overline{M} \oplus M^{\perp}$. Also, it should be noticed that $M \subseteq \overline{M} \subseteq M^{\perp\perp}$.

Now, let $x \in M^{\perp\perp}$. From $H = \overline{M} \oplus M^{\perp}$, it follows that we can write $x = u + v$ with $u \in \overline{M}$ and $v \in M^{\perp}$. This implies $x - u = v \in M^{\perp}$ and since $u \in \overline{M} \subseteq M^{\perp\perp}$, we have $x - u \in M^{\perp\perp}$. Hence, $x - u \in M^{\perp} \cap M^{\perp\perp} = \{0\}$ or $x = u \in \overline{M}$. Therefore, $M^{\perp\perp} \subseteq \overline{M}$ also holds true, and so $M^{\perp\perp} = \overline{M}$, as desired.

Problem 33.9. *For two arbitrary vector subspaces M and N of a Hilbert space establish the following:*

a. $(M + N)^{\perp} = M^{\perp} \cap N^{\perp}$, *and*
b. *if M and N are both closed, then* $(M \cap N)^{\perp} = \overline{M^{\perp} + N^{\perp}}$.

Solution. (a) Let $x \perp M + N$. Then, $x \perp y$ holds for all $y \in M$ and $x \perp z$ holds for all $z \in N$. That is, $x \in M^{\perp}$ and $x \in N^{\perp}$. Therefore, $(M + N)^{\perp} \subseteq M^{\perp} \cap N^{\perp}$.

For the reverse inclusion, let $x \in M^{\perp} \cap N^{\perp}$ and let $y \in M + N$. Write $y = u + v$ with $u \in M$ and $v \in N$ and note that $(x, y) = (x, u) + (x, v) = 0$ holds. Hence, $x \in (M + N)^{\perp}$, and therefore $M^{\perp} \cap N^{\perp} \subseteq (M + N)^{\perp}$. Thus, $(M + N)^{\perp} = M^{\perp} \cap N^{\perp}$.

(b) Now, suppose that M and N are closed subspaces. By the preceding problem we know that $M = M^{\perp\perp}$ and $N = N^{\perp\perp}$. Now, use part (a) to get

$$(M^{\perp} + N^{\perp})^{\perp} = M \cap N.$$

Therefore,

$$(M \cap N)^{\perp} = \left[(M^{\perp} + N^{\perp})^{\perp}\right]^{\perp} = \overline{M^{\perp} + N^{\perp}}.$$

Problem 33.10. *Let X be an inner product space such that $M = M^{\perp\perp}$ holds for every closed subspace M. Show that X is a Hilbert space.*

Solution. We need to show that X is complete in the induced norm. For this, it suffices to establish that $X = \hat{X}$, where $\hat{X}$ denotes the norm completion of X. (We already know that $\hat{X}$ is a Hilbert space; see Problem 33.5.) To this end, let $\hat{u} \in \hat{X}$ be a nonzero vector.

The linear functional $f\colon \hat{X} \to \mathbf{C}$ defined by $f(x) = (x, \hat{u})$ is nonzero and continuous. So, f restricted to X is also continuous and since X is norm dense in $\hat{X}$, it follows that $f\colon X \to \mathbf{C}$ is a nonzero continuous linear functional. In particular, its kernel $M = \{x \in X\colon f(x) = 0\}$ is a proper closed subspace of X. We claim that $M^{\perp} \neq \{0\}$. Indeed, if $M^{\perp} = \{0\}$, then it follows from our hypothesis that $M = M^{\perp\perp} = \{0\}^{\perp} = X$, which is a contradiction.

Next, fix a vector $u \in M^{\perp}$ with $\|u\| = 1$ and let $v = \overline{f(u)}u \in X$. Now, taking into account that $f(x)u - f(u)x \in M$ holds for each $x \in X$, it follows that

$$f(x) = f(x)(u, u) = f(u)(x, u) = (x, v).$$

for each $x \in X$. That is, $(x, \hat{u}) = (x, v)$ for all $x \in X$. Since X is dense in $\hat{X}$, we get $(x, v) = (x, \hat{u})$ for all $x \in \hat{X}$. That is, $(x, v - \hat{u}) = 0$ for all $x \in \hat{X}$, and from this we conclude that $\hat{u} = v \in X$. So, $X = \hat{X}$, and thus X is a Hilbert space.

Problem 33.11. *Consider the linear operator $V: L_2[a, b] \to L_2[a, b]$ defined by*

$$Vf(x) = \int_a^x f(t)\,dt.$$

Show that the norm of the operator satisfies $\|V\| \leq b - a$.

Solution. By Hölder's Inequality, we get

$$\begin{aligned}
|Vf(x)| &\leq \int_a^x |f(t)|\,dt \leq \int_a^b |f(t)|\,dt \\
&\leq \Big[\int_a^b |f(t)|^2\,dt\Big]^{\frac{1}{2}} \Big[\int_a^b 1^2\,dt\Big]^{\frac{1}{2}} \\
&\leq (b-a)^{\frac{1}{2}} \|f\|.
\end{aligned}$$

Therefore, the norm of V satisfies

$$\begin{aligned}
\|Vf\|^2 &= \int_a^b |Vf(t)|^2\,dt \leq (b-a)\int_a^b \|f\|^2\,dt \\
&\leq (b-a)^2\|f\|^2.
\end{aligned}$$

This implies $\|V\| \leq b - a$.

Problem 33.12. *Let $\{x_n\}$ be a norm bounded sequence of vectors in the Hilbert space ℓ_2, where $x_n = (x_1^n, x_2^n, x_3^n, \ldots)$. If for each fixed coordinate k we have $\lim_{n\to\infty} x_k^n = 0$, then show that*

$$\lim_{n\to\infty} (x_n, y) = 0$$

holds for each vector $y \in \ell_2$.

Solution. Choose some $\lambda > 0$ such that $\|x_n\| \leq \lambda$ holds for all n. Now, fix a vector $y = (y_1, y_2, \ldots) \in \ell_2$ and let $\epsilon > 0$. Pick some m satisfying $\left(\sum_{k=m}^{\infty} |y_k|^2\right)^{\frac{1}{2}} < \epsilon$.

Since for each fixed k we have $\lim_{n\to\infty} x_k^n = 0$, there exists an integer n_0 satisfying $\left|\sum_{k=1}^{m} x_k^n \overline{y_k}\right| < \epsilon$ for all $n \geq n_0$. Now, using the Cauchy–Schwarz inequality, we see that for each $n \geq n_0$ we have

$$\begin{aligned}
|(x_n, y)| = \left|\sum_{k=1}^{\infty} x_k^n \overline{y_k}\right| &\leq \left|\sum_{k=1}^{m} x_k^n \overline{y_k}\right| + \sum_{k=m+1}^{\infty} |x_k^n \overline{y_k}| \\
&< \epsilon + \Big(\sum_{k=m+1}^{\infty} |x_k^n|^2\Big)^{\frac{1}{2}} \Big(\sum_{k=m+1}^{\infty} |y_k|^2\Big)^{\frac{1}{2}} \\
&\leq \epsilon + \lambda\epsilon = (1+\lambda)\epsilon .
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we have shown that $\lim_{n\to\infty}(x_n, y) = 0$.

Problem 33.13. *Let H be a Hilbert space and let $\{x_n\}$ be a sequence satisfying*

$$\lim_{n\to\infty} (x_n, y) = (x, y)$$

for each $y \in H$. Show that there exists a subsequence $\{x_{k_n}\}$ of $\{x_n\}$ such that

$$\lim_{n\to\infty} \left\| \frac{x_{k_1} + x_{k_2} + x_{k_3} + \cdots + x_{k_n}}{n} - x \right\| = 0.$$

Solution. Start by noticing that we can assume without loss of generality that $x = 0$. Therefore, suppose

$$\lim_{n\to\infty} (x_n, y) = 0$$

for each $y \in X$. We claim that the sequence $\{x_n\}$ is norm bounded. To see this, for each n consider the continuous linear functional $f_n \colon H \to \mathbb{C}$ defined by $f_n(y) = (y, x_n)$ for each $y \in H$. By our condition, the sequence of bounded linear functionals $\{f_n\}$ is pointwise bounded. So, by the Principle of Uniform Boundedness (Theorem 28.8), there exists some $C > 0$ such that $\|f_n\| \leq C$ for each n. Now, notice that (by Theorem 33.9) $\|x_n\| = \|f_n\|$ holds for each n.

Now, let $k_1 = 1$ and then choose $k_2 > k_1$ with $|(x_{k_1}, x_{k_2})| < 1$. Next, an inductive argument shows that there exist integers $k_1 < k_2 < k_3 < \cdots < k_n < k_{n+1} < \cdots$ satisfying

$$|(x_{k_i}, x_{k_{n+1}})| < 2^{-n} \quad \text{for each } 1 \leq i \leq n.$$

To finish the solution, we shall show that

$$\left\| \frac{x_{k_1}+x_{k_2}+x_{k_3}+\cdots+x_{k_n}}{n} \right\|^2 \leq \frac{2+C^2}{n}$$

holds for each n. To see this, take inner products to get

$$\begin{aligned}\left\| \frac{\sum_{i=1}^{n} x_{k_i}}{n} \right\|^2 &= \frac{\sum_{i=1}^{n}(x_{k_1}, x_{k_i}) + \sum_{i=1}^{n}(x_{k_2}, x_{k_i}) + \cdots + \sum_{i=1}^{n}(x_{k_n}, x_{k_i})}{n^2} \\ &\leq \frac{n[C^2+(1+2^{-1}+2^{-2}+2^{-3}+\cdots+2^{-n})]}{n^2} \leq \frac{2+C^2}{n}.\end{aligned}$$

This implies

$$\lim_{n\to\infty} \left\| \frac{x_{k_1}+x_{k_2}+x_{k_3}+\cdots+x_{k_n}}{n} \right\| = 0,$$

as required.

Problem 33.14. *Let $\rho\colon [a,b] \to (0,\infty)$ be a measurable essentially bounded function and for each $n = 0, 1, 2, \ldots$ let P_n be a nonzero polynomial of degree n. Assume that*

$$\int_a^b \rho(x)P_n(x)\overline{P_m(x)}\,dx = 0 \text{ for } n \neq m.$$

Show that each P_n has n distinct real roots all lying in the open interval (a,b).

Solution. By Theorem 33.12, we know that the sequence of orthogonal polynomials $P_0, P_1, P_2, \ldots$ is complete and coincides (aside of scalar factors) with the sequence of orthogonal functions of $L_2(\rho)$ that is obtained by applying the Gram–Schmidt orthogonalization process to the sequence of linearly independent functions $\{1, x, x^2, x^3, \ldots\}$. In particular, we have $\int_a^b \rho(x)x^m P_n(x)\,dx = 0$ for all $m = 0, 1, \ldots, n-1$. Also, by multiplying each P_n by an appropriate scalar, we can assume that each P_n has real coefficients and leading coefficient 1.

Now, fix one of these polynomials P_n, where $n \geq 1$. First, we shall show that P_n cannot have any complex roots. If P_n has a complex root, then P_n has a factorization of the form $P_n(x) = [(x+\alpha)^2+\beta^2]Q(x)$, where α and β are real numbers and Q is a polynomial of degree $n-2$. Hence $Q(x)P_n(x) = [(x+\alpha)^2+\beta^2][Q(x)]^2 \geq 0$,

and from $\int_a^b \rho(x)x^m P_n(x)\,dx = 0$ for all $m = 0, 1, \ldots, n-1$, it follows that

$$0 < \int_a^b \rho(x)Q(x)P_n(x)\,dx = 0,$$

which is impossible. Hence, each P_n has only real roots. This means that P_n has a factorization of the form

$$P_n(x) = (x - r_1)^{m_1}(x - r_2)^{m_2} \cdots (x - r_k)^{m_k},$$

where $r_1, r_2, \ldots, r_k$ are real number and $m_1, m_2, \ldots, m_k$ are natural numbers such that $m_1 + m_2 + \cdots + m_k = n$.

Next, we claim that P_n does not have any root outside of the open interval (a, b). To see this, assume that one root lies outside of (a, b), say $r_1 \leq a$. Then the polynomial $Q(x) = (x - r_2)^{m_2} \cdots (x - r_k)^{m_k}$ has degree less than n and satisfies $Q(x)P_n(x) \geq 0$ for each $a \leq x \leq b$. But then, we have

$$0 < \int_a^b \rho(x)Q(x)P_n(x)\,dx = 0,$$

which is a contradiction.

Finally, to see that each root appears with multiplicity one, assume by way of contradiction that one root has multiplicity more than one, say $m_1 > 1$. If, again

$$Q(x) = \begin{cases} (x - r_2)^{m_2} \cdots (x - r_k)^{m_k} & \text{if } m_1 \text{ is even} \\ (x - r_1)(x - r_2)^{m_2} \cdots (x - r_k)^{m_k} & \text{if } m_1 \text{ is odd,} \end{cases}$$

then Q is a polynomial of degree strictly less than n and satisfies $Q(x)P_n(x) \geq 0$ for all $a \leq x \leq b$. But then, as previously,

$$0 = \int_a^b \rho(x)Q(x)P_n(x)\,dx > 0,$$

which is absurd. Hence, each polynomial P_n has n distinct real roots all lying in the open interval (a, b).

Problem 33.15. *In Example 33.13 we defined the sequence $P_0, P_1, P_2, \ldots$ of Legendre polynomials by the formulas*

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2 - 1)^n.$$

We also proved that these are (aside of scalar factors) the polynomials obtained by applying the Gram–Schmidt orthogonalization process to the sequence of linearly independent functions $\{1, x, x^2, \dots\}$ in the Hilbert space $L_2([-1,1])$. Show that for each n we have

$$P_n(1) = 1 \quad \textit{and} \quad \|P_n\| = \sqrt{\tfrac{2}{2n+1}}.$$

Solution. The proof of the formula $P_n(1) = 1$ is by induction. Notice that for $n = 0$ and $n = 1$ the formula is trivially true. So, for the induction argument, assume that $P_n(1) = 1$ holds true for some n. To complete the proof, we must show that $P_{n+1}(1) = 1$. To see this, note that

$$\begin{aligned}
P_{n+1}(x) &= \frac{1}{2^{n+1}(n+1)!}\frac{d^{n+1}}{dx^{n+1}}(x^2-1)^{n+1} \\
&= \frac{1}{2^{n+1}(n+1)!}\frac{d^n}{dx^n}\Big[\big((x^2-1)^{n+1}\big)'\Big] \\
&= \frac{1}{2^{n+1}(n+1)!}\frac{d^n}{dx^n}\big[2(n+1)x(x^2-1)^n\big] \\
&= \frac{1}{2^n n!}\frac{d^n}{dx^n}\big[(x-1)(x^2-1)^n\big] + \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n \\
&= (x-1)Q(x) + P_n(x),
\end{aligned}$$

where the term $(x-1)Q(x)$ designates the form of the expression

$$\frac{1}{2^n n!}\frac{d^n}{dx^n}(x-1)(x^2-1)^n = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x-1)^{n+1}(x+1)^n.$$

So, $P_{n+1}(1) = P_n(1) = 1$.

Next, we shall compute the norm of P_n. Clearly,

$$\begin{aligned}
\|P_n\|^2 &= \int_{-1}^{1} P_n(x)\,P_n(x)\,dx \\
&= \frac{1}{(2^n n!)^2}\int_{-1}^{1}\frac{d^n}{dx^n}(x^2-1)^n\,\frac{d^n}{dx^n}(x^2-1)^n\,dx.
\end{aligned}$$

Next, observe that the function $(x^2-1)^n = (x-1)^n(x+1)^n$ and all of its derivatives of order less than or equal to $n-1$ vanish at the points ± 1. So, integrating by

parts n-times and using the previous observation, we obtain

$$\|P_n\|^2 = \frac{(-1)^n}{(2^n n!)^2}\int_{-1}^{1}(x^2-1)^n \frac{d^{2n}}{dx^{2n}}(x^2-1)^n\,dx = \frac{(-1)^n}{(2^n n!)^2}\int_{-1}^{1}(x^2-1)^n(2n)!\,dx$$
$$= \frac{(-1)^n(2n)!}{(2^n n!)^2}\int_{-1}^{1}(x^2-1)^n\,dx = \frac{(2n)!}{(2^n n!)^2}\int_{-1}^{1}(1-x^2)^n\,dx.$$

Using integration by parts to evaluate this last integral gives

$$\begin{aligned}
\int_{-1}^{1}(1-x^2)^n\,dx &= \int_{-1}^{1}(1-x)^n(1+x)^n\,dx\\
&= \tfrac{(1-x)^n(1+x)^{n+1}}{n+1}\Big|_{-1}^{1} + \int_{-1}^{1} n(1-x)^{n-1}\tfrac{(1+x)^{n+1}}{n+1}\,dx\\
&= \frac{n}{n+1}\int_{-1}^{1}(1-x)^{n-1}(1+x)^{n+1}\,dx = \cdots\\
&= \frac{n!}{(n+1)\cdots(2n)}\int_{-1}^{1}(1+x)^{2n}\,dx\\
&= \frac{(n!)^2\,2^{2n+1}}{(2n)!\,(2n+1)}.
\end{aligned}$$

Consequently, the norm of P_n is given by

$$\|P_n\|^2 = \left[\frac{(2n)!}{(2^n n!)^2}\right]\left[\frac{(n!)^2\,2^{2n+1}}{(2n)!\,(2n+1)}\right] = \frac{2}{2n+1},$$

as claimed.

Problem 33.16. *Let $\{T_\alpha\}_{\alpha\in A}$ be a family of linear continuous operators from a complex Hilbert space X into another complex Hilbert space Y. Assume that for each $x \in X$ and each $y \in Y$ the set of complex numbers $\{(T_\alpha(x), y)\colon \alpha \in A\}$ is bounded. Show that the family of operators $\{T_\alpha\}_{\alpha\in A}$ is uniformly norm bounded, i.e., show that there exists some constant $M > 0$ satisfying $\|T_\alpha\| \le M$ for all $\alpha \in A$.*

Solution. Observe that if Z is a Banach space over the field of complex numbers, then we may also consider Z as a Banach space over the field of real numbers. Therefore, the Principle of Uniform Boundedness (Theorem 28.8) can be applied to any Banach space over the field of complex numbers.

Fix a vector $x \in X$. For each $\alpha \in A$ define the complex valued continuous linear operator $B_\alpha: Y \to \mathbf{C}$ by

$$B_\alpha(y) = \big(y, T_\alpha(x)\big).$$

Thus, $\{B_\alpha\}_{\alpha \in A}$ is family of bounded linear operators from the Banach space Y to the Banach space of complex numbers C. From Theorem 33.9, we know that

$$\|B_\alpha\| = \|T_\alpha(x)\|.$$

Next, notice that for each fixed $y \in Y$, it follows from

$$|B_\alpha(y)| = |(y, T_\alpha(x))| = |(T_\alpha(x), y)|$$

and our hypothesis that the family of continuous linear operators $\{B_\alpha\}_{\alpha \in A}$ is pointwise bounded. Hence, by the Principle of Uniform Boundedness (Theorem 28.8), the family $\{B_\alpha\}_{\alpha \in A}$ is norm bounded. This means that there exists a constant $M_x > 0$ (that depends upon x) such that $\|B_\alpha\| \le M_x$ for all $\alpha \in A$. Thus, we have $\|T_\alpha(x)\| \le M_x$ for all α.

Therefore, the family $\{T_\alpha\}_{\alpha \in A}$ of continuous linear operators $T_\alpha: X \to Y$ is pointwise bounded. Invoking the Principle of Uniform Boundedness once more, we conclude that there exists a constant $M > 0$ satisfying $\|T_\alpha\| \le M$ for all $\alpha \in A$.

Problem 33.17. *Let $\{\phi_n\}$ be an orthonormal sequence in a Hilbert space H and consider the operator $T: H \to H$ defined by*

$$T(x) = \sum_{n=1}^{\infty} \alpha_n (x, \phi_n)\phi_n,$$

where $\{\alpha_n\}$ is a sequence of scalars satisfying $\lim \alpha_n = 0$. *Show that T is a compact operator.*

Solution. Let B be the open unit ball of H. We need to show that $\overline{T(B)}$ is a compact set. For this, it suffices to show that $T(B)$ is totally bounded. To this end, fix $\epsilon > 0$ and observe that there exists an integer m such that $|\alpha_n| < \epsilon$ holds for all $n > m$.

Next, define the operator $T_m: H \to H$ by

$$T_m(x) = \sum_{i=1}^{m} \alpha_i (x, \phi_i)\phi_i.$$

Clearly, the range of T_m is a finite dimensional subspace of H and thus, T_m is a compact operator. Therefore, $T_m(B)$ is a totally bounded set. Thus, there exists a finite set $\{y_1, y_2, \ldots, y_n\}$ such that for each $x \in B$ there exists some $1 \le k \le n$ such that $\|T_m(x) - y_k\| < \epsilon$. Now, Parseval's Inequality $\sum_{i=1}^{\infty} |(x, \phi_i)|^2 \le \|x\|^2 < 1$ implies

$$\|T(x) - T_m(x)\|^2 = \Big\| \sum_{i=m+1}^{\infty} \alpha_i (x, \phi_i)\phi_i \Big\|^2 = \sum_{i=m+1}^{\infty} |\alpha_i|^2 \, |(x, \phi_i)|^2$$
$$\le \epsilon^2 \sum_{i=m+1}^{\infty} |(x, \phi_i)|^2 \le \epsilon^2 \|x\|^2 < \epsilon^2.$$

Therefore, for each $x \in B$ there exists some $1 \le k \le n$ such that

$$\|T(x) - y_k\| \le \|T(x) - T_m(x)\| + \|T_m(x) - y_k\| \le \epsilon + \epsilon = 2\epsilon.$$

This shows that $T(B)$ is totally bounded, and hence T is a compact operator.

Problem 33.18. *Assume that $T, T^*: H \to H$ are two functions on a Hilbert space satisfying*

$$(Tx, y) = (x, T^*y)$$

for all $x, y \in H$. Show that T and T^* are both bounded linear operators satisfying

$$\|T\| = \|T^*\| \quad \textit{and} \quad \|TT^*\| = \|T\|^2.$$

Solution. By the symmetry of the situation, it suffices to show that T is a bounded linear operator. We shall show first that T is a linear operator. To this end, fix $x, y \in H$ and two scalars α and β. Then, for each $z \in H$ we have

$$\big(T(\alpha x + \beta y), z\big) = \big(\alpha x + \beta y, T^*z\big) = \alpha(x, T^*z) + \beta(y, T^*z)$$
$$= \alpha(Tx, z) + \beta(Ty, z) = \big(\alpha Tx + \beta Ty, z\big),$$

or $\big(T(\alpha x + \beta y) - \alpha Tx - \beta Ty, z\big) = 0$ for all $z \in H$. This implies (by letting $z = T(\alpha x + \beta y) - \alpha Tx - \beta Ty$) that

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y),$$

i.e., that T is a linear operator.

Next, for each $y \in H$ with $\|y\| \leq 1$ consider the linear functional $f_y: H \to H$ defined by $f_y(x) = (x, T^*y)$. Then, using the Cauchy–Schwarz Inequality, we see that

$$|f_y(x)| = |(x, T^*(y)| = |(T(x), y)| \leq \|T(x)\| \|y\| \leq \|T(x)\|$$

for all $y \in H$ with $\|y\| \leq 1$. This implies that the set of linear functionals $\{f_y: \|y\| \leq 1\}$ is pointwise bounded and therefore, by the Principle of Uniform Boundedness, the set of linear functionals $\{f_y: \|y\| \leq 1\}$ is norm bounded. Thus, there exists some constant $C > 0$ such that $\|f_y\| \leq C$ for all $y \in H$ with $\|y\| \leq 1$.

Now, assume that $y \in H$ satisfies $\|y\| \leq 1$ and $T^*(y) \neq 0$. Letting $x = T^*(y)/\|T^*(y)\|$, we obtain

$$\|T^*(y)\| = \frac{1}{\|T^*(y)\|}\left|\left(T^*(y), T^*(y)\right)\right| = \left|\left(x, T^*(y)\right)\right| = |f_y(x)| \leq C.$$

This implies

$$\|T^*\| = \sup\left\{\|T^*(y)\|: \|y\| \leq 1\right\} \leq C,$$

and thus, T^* is a bounded linear operator. By the symmetry of the situation, T is likewise a bounded linear operator.

Now, note that for all $x, y \in H$ with $\|x\| = \|y\| = 1$, we have

$$|(Tx, y)| = |(x, T^*y)| \leq \|x\| \|T^*y\| \leq \|x\| \|T^*\| \|y\| = \|T^*\|,$$

and hence, $\|Tx\| = \sup\left\{|(Tx, y)|: \|y\| \leq 1\right\} \leq \|T^*\|$ for all unit vectors $x \in H$. This implies

$$\|T\| = \sup_{\|x\|=1} \|T(x)\| \leq \|T^*\|.$$

Using the symmetry once more, we get $\|T^*\| \leq \|T\|$, and so $\|T\| = \|T^*\|$.

Finally, for each $x \in H$ with $\|x\| = 1$, we have

$$\|TT^*x\| \leq \|T\| \|T^*\| \|x\| = \|T\|^2, \text{ and}$$
$$\|T^*x\|^2 = (T^*x, T^*x) = (TT^*x, x) \leq \|TT^*\| \|x\| \|x\| = \|TT^*\|,$$

and so by taking suprema, we get $\|TT^*\| = \|T\|^2$.

Problem 33.19. *Show that if $T: H \to H$ is a bounded linear operator on a Hilbert space, then there exists a unique bounded operator $T^*: H \to H$ (called*

the **adjoint operator** *of T) satisfying*

$$(Tx, y) = (x, T^*y)$$

for all $x, y \in H$. Moreover, show that $\|T\| = \|T^*\|$.

Solution. Assume that $T: H \to H$ is a bounded linear operator on a Hilbert space. For each fixed $y \in H$, the formula $f_y(x) = (Tx, y)$ defines a bounded linear functional on H. By Theorem 33.9, there exists a unique vector $T^*y \in H$ satisfying

$$f_y(x) = (Tx, y) = (x, T^*y)$$

for all $x \in H$. Now, use the preceding problem to conclude that the unique function $T^*: H \to H$ defined above is, in fact, a bounded linear operator satisfying $\|T^*\| = \|T\|$.

34. ORTHONORMAL BASES

Problem 34.1. *Let $\{e_i\}_{i\in I}$ and $\{f_j\}_{j\in J}$ be two orthonormal bases of a Hilbert space. Show that I and J have the same cardinality.*

Solution. Assume that $\{e_i\}_{i\in I}$ and $\{f_j\}_{j\in J}$ are two orthonormal bases of a Hilbert space. First, suppose that I is a finite set. Then, from Theorem 34.2, it follows that $\{e_i\}_{i\in I}$ is also a Hamel basis and so H is finite dimensional. Since the f_j (as being mutually orthogonal vectors) are also linearly independent, we conclude that J must also be a finite set. This implies that $\{f_j\}_{j\in J}$ is itself a Hamel basis for H, and so I and J must have the same number of elements.

Now, suppose that I and J are infinite sets. For each $i \in I$, we define the set of indices $J_i = \{j \in J: (e_i, f_j) \neq 0\}$. By Theorem 34.2, we know that each J_i is nonempty and at-most countable. Next, we claim that

$$J = \bigcup_{i\in I} J_i. \qquad (\star)$$

To see this, let $j \in J$. Since $\{e_i\}_{i\in I}$ is an orthonormal basis, it follows from Parseval's Identity that $\sum_{i\in I} |(f_j, e_i)|^2 = \|f_j\|^2 = 1$, and so $(e_i, f_j) \neq 0$ holds true for some $i \in I$. Thus, $j \in J_i$ holds true for at least one $i \in I$, and thus $J = \bigcup_{i\in I} J_i$.

Finally, to see that I and J have the same cardinality use ($\star$) together with the standard "cardinality" arithmetic; see, for instance, P. R. Halmos, *Naive Set Theory*, Springer–Verlag, 1974, pp. 94–98.

Problem 34.2. *Let $\{e_i\}_{i\in I}$ be an orthonormal basis in a Hilbert space H. If D is a dense subset of H, then show that the cardinality of D is at least as large as that of I. Use this conclusion to provide an alternate proof of Theorem 34.4 by proving that for an infinite dimensional Hilbert space H the following statements are equivalent:*

1. *H has a countable orthonormal basis.*
2. *H is separable.*
3. *H is linearly isometric to ℓ_2.*

Solution. Let $\{e_i\}_{i\in I}$ be an orthonormal basis in a Hilbert space H and let D be a dense subset of H. Consider the family of open balls $\{B(e_i, \frac{1}{2})\}_{i\in I}$, where

$$B(e_i, \tfrac{1}{2}) = \{x \in H\colon \|e_i - x\| < \tfrac{1}{2}\}.$$

Since $\|e_i - e_j\| = \sqrt{2}$ for $i \neq j$, it follows that $\{B(e_i, \frac{1}{2})\}_{i\in I}$ is a pairwise disjoint family of open sets. Since D is dense in H for each $i \in I$, there exists some $d_i \in D \cap B(e_i, \frac{1}{2})$. Clearly, the mapping $i \mapsto d_i$, from I into D, is one-to-one and this shows that D has cardinality greater than or equal of the cardinality of I.

Next, we shall prove the equivalent statements. To this end, assume that H is an infinite dimensional Hilbert space.

(1) $\Longleftrightarrow$ (2) Let $\{e_1, e_2, \ldots\}$ be a countable orthonormal basis for H. Then, the finite linear combinations of the e_n with "rational" coefficients is a countable dense set.

Now, assume that H is separable, and let D be a countable dense subset of H. If $\{e_i\}_{i\in I}$ is an orthonormal basis, it follows form the first part that I has cardinality at most that of D, and hence, I is at-most countable. Since H is infinite dimensional, I must be countable, and so H has a countable orthonormal basis.

(2) $\Longrightarrow$ (3) If H has a countable orthonormal basis, then H is linearly isometric (by Theorem 34.9) to $\ell_2(\mathbb{N}) = \ell_2$.

(3) $\Longrightarrow$ (1) Obvious.

Problem 34.3. *Let I be an arbitrary nonempty set and for each $i \in I$ let $e_i = \chi_{\{i\}}$. Show that the family of functions $\{e_i\}_{i\in I}$ is an orthonormal basis for the Hilbert space $\ell_2(I)$.*

Solution. For each i, let $e_i = \chi_{\{i\}}$ and note that the family of functions $\{e_i\}_{i\in I}$ is an orthonormal family. Now, notice that if $x = \{x_i\}_{i\in I} \in \ell_2(I)$, then

$$\|x\|^2 = \sum_{i\in I} |x_i|^2 = \sum_{i\in I} |(x, e_i)|^2.$$

This shows that $\{e_i\}_{i\in I}$ is an orthonormal basis for the Hilbert space $\ell_2(I)$.

Problem 34.4. *Let $\{e_i\}_{i\in I}$ be an orthonormal basis in a Hilbert space and let x be a unit vector, i.e, $\|x\| = 1$. Show that for each $k \in \mathbb{N}$ the set $\{i \in I\colon |(x, e_i)| \geq \frac{1}{k}\}$ has at most k^2 elements.*

Solution. From Parseval's Identity we know that

$$1 = \|x\|^2 = \sum_{i\in I} |(x, e_i)|^2.$$

Let $A = \{i \in I\colon |(x, e_i)| \geq \frac{1}{k}\}$. If A has more than k^2 elements, then by choosing $k^2 + 1$ indices from A, we see that

$$1 = \|x\|^2 = \sum_{i\in I} |(x, e_i)|^2 \geq (k^2 + 1)\tfrac{1}{k^2} > 1,$$

which is impossible. Therefore, A has at most k^2 elements.

Problem 34.5. *Let M be a closed vector subspace of a Hilbert space H and let $\{e_i\}_{i\in I}$ be an orthonormal basis of M; where M is now considered as a Hilbert space in its own right under the induced operations. If $x \in H$, then show that the unique vector of M closest to x (which is guaranteed by Theorem 33.6) is the vector $y = \sum_{i\in I}(x, e_i)e_i$.*

Solution. Assume that $\{e_i\}_{i\in I}$ is an orthonormal basis for a closed subspace M of a Hilbert space H and let $x \in H$ be a fixed vector. Note first that Parseval's Inequality guarantees that $\sum_{i\in I} |(x, e_i)|^2 \leq \|x\|^2$, and so $y = \sum_{i\in I}(x, e_i)e_i$ is a well-defined vector of M.

We claim that $x - y \perp M$. To see this, let z be an arbitrary vector of M, and let $z = \sum_{j\in I}(z, e_j)e_j$ be its Fourier series expansion as a vector of M. Then, we have

$$\begin{aligned}(z, x - y) &= \Big(\sum_{j\in I}(z, e_j)e_j, x - \sum_{i\in I}(x, e_i)e_i\Big)\\ &= \sum_{j\in I}(z, e_j)(e_j, x) - \sum_{j\in I}\sum_{i\in I}\big((z, e_j)e_j, (x, e_i)e_i\big)\\ &= \sum_{j\in I}(z, e_j)(e_j, x) - \sum_{j\in I}(z, e_j)(e_j, x) = 0.\end{aligned}$$

Now, if z is an arbitrary vector of M, then $y - z \in M$ and so $x - y \perp y - z$. Hence, by the Pythagorean Theorem,

$$\|x - z\|^2 = \|(x - y) + (y - z)\|^2 = \|x - y\|^2 + \|y - z\|^2 \geq \|x - y\|^2.$$

This shows that y is the vector in M closest to x.

Problem 34.6. *Let $\{e_n\}$ be an orthonormal basis of a separable Hilbert space. For each n let $f_n = e_{n+1} - e_n$. Show that the vector subspace generated by the sequence $\{f_n\}$ is dense.*

Solution. We need to show that if $x \perp f_n$ for all n, then $x = 0$. So, let x be a vector satisfying $x \perp (e_{n+1} - e_n)$ for each n. That is,

$$0 = (x, e_{n+1} - e_n) = (x, e_{n+1}) - (x, e_n).$$

This implies $(x, e_{n+1}) = (x, e_n)$ for each n. If we let $\delta = (x, e_1)$, then $\delta = (x, e_n)$ for all n, and by Parseval's Identity

$$\|x\|^2 = \sum_{n=1}^{\infty} |(x, e_n)|^2 = \sum_{n=1}^{\infty} \delta^2$$

Therefore, $\delta = 0$, and hence, $\|x\| = 0$, or $x = 0$.

Problem 34.7. *Show that a linear operator $L\colon H_1 \to H_2$ between two Hilbert spaces is norm preserving if and only if it is inner product preserving.*

Solution. Let $L\colon H_1 \to H_2$ be a linear operator between two Hilbert spaces. If L is inner product preserving, then

$$\|Lx\|^2 = (Lx, Lx) = (x, x) = \|x\|^2$$

holds or $\|Lx\| = \|x\|$ for each $x \in H_1$, i.e., L is norm preserving. For the converse, assume that L is norm preserving. Then, from Theorem 32.6, it follows that

$$\begin{aligned}
(Lx, Ly) &= \tfrac{1}{4}\big(\|Lx + Ly\|^2 - \|Lx - Ly\|^2 + \imath\|Lx + \imath Ly\|^2 - \imath\|Lx - \imath Ly\|^2\big) \\
&= \tfrac{1}{4}\big(\|L(x + y)\|^2 - \|L(x - y)\|^2 + \imath\|L(x + \imath y)\|^2 - \imath\|L(x - \imath y)\|^2\big) \\
&= \tfrac{1}{4}\big(\|x + y\|^2 - \|x - y\|^2 + \imath\|x + \imath y\|^2 - \imath\|x - \imath y\|^2\big) \\
&= (x, y).
\end{aligned}$$

That is, L is inner product preserving.

Problem 34.8. *Show that the vector space $\ell_2(Q)$ of all square summable complex-valued functions defined on a nonempty set Q under the inner product*

$$(x, y) = \sum_{q \in Q} x(q)\overline{y(q)}$$

is a Hilbert space.

Solution. The verification of the inner product properties of the function $(\cdot, \cdot)$ are straightforward. We shall show that $\ell_2(Q)$ is a Hilbert space, i.e., complete under its induced normed.

To see this, assume that Q is an infinite set, and let $\{x_n\} \subseteq \ell_2(Q)$ be a Cauchy sequence. Since for each n we have $x_n(q) \neq 0$ for at-most countably many $q \in Q$, there exists an at-most countable subset C of Q such that $x_n(q) = 0$ for all $q \in Q \setminus C$ and all n. We consider only the case when C is a countable set, say $C = \{q_1, q_2, \ldots\}$. For each n, let

$$y_n = \big(x_n(q_1), x_n(q_2), \ldots\big).$$

Then, it is easy to see that we can consider $\{y_n\}$ as a Cauchy sequence in ℓ_2. The completeness of ℓ_2 implies that $\{y_n\}$ converges to some sequence $y = (y_1, y_2, \ldots)$ in ℓ_2. If $x\colon Q \to \mathbb{C}$ is defined by $x(q_i) = y_i$ and $x(q) = 0$ whenever $q \in Q \setminus C$, then $x \in \ell_2(Q)$ and $\|x_n - x\| = \|y_n - y\| \to 0$ holds in $\ell_2(Q)$. This shows that $\ell_2(Q)$ is a Hilbert space.

Problem 34.9. *Let $\{e_i\}_{i\in I}$ be an orthonormal basis of a Hilbert space H. Show that the linear operator $L\colon H \to \ell_2(I)$, defined by*

$$L(x) = \{(x, e_i)\}_{i\in I},$$

is a surjective linear isometry.

Solution. Clearly, L is linear and by Parseval's Identity (Theorem 34.2(5)), it is also an isometry. We shall verify next that L is also surjective. To this end, let $\{\lambda_i\}_{i\in I} \in \ell_2(I)$. From $\sum_{i\in I} |\lambda_i|^2 < \infty$, it follows that $\lambda_i \neq 0$ for at-most countably many indices i. Assume that $\{i \in I\colon \lambda_i \neq 0\} = \{\lambda_{i_1}, \lambda_{i_2}, \ldots\}$. (We consider only the countable case; the finite case is trivial.) Clearly, $\sum_{n=1}^{\infty} |\lambda_{i_n}|^2 < \infty$.

From the Pythagorean Theorem, we have

$$\Big\|\sum_{k=n}^{m} \lambda_{i_k} e_{i_k}\Big\|^2 = \sum_{k=n}^{m} |\lambda_{i_k}|^2,$$

and from this it follows that the series $\sum_{n=1}^{\infty} \lambda_{i_n} e_{i_n}$ is norm convergent in H. Let $x = \sum_{n=1}^{\infty} \lambda_{i_n} e_{i_n} = \sum_{i\in I} \lambda_i e_i$. Then, $(x, e_i) = \lambda_i$ for each, i and so $L(x) = \{\lambda_i\}_{i\in I}$. This shows that L is also surjective, as required.

Problem 34.10. *Let $\{e_n\}$ be an orthonormal sequence of vectors in the Hilbert space $L_2[0, 2\pi]$. Suppose that for each continuous function f in $L_2[0, 2\pi]$ we have $f = \sum_{n=1}^{\infty}(f, e_n)e_n$. Show that $\{e_n\}$ is an orthonormal basis.*

Solution. We need only show that the linear span of the set $\{e_n\}$ is dense. Let $\epsilon > 0$ and let $g \in L_2[0, 2\pi]$. Since the continuous functions are dense in $L_2[0, 2\pi]$ (see Theorem 31.10), there exists a continuous function $f \in L_2[0, 2\pi]$ with $\|f - g\| \leq \epsilon$. By our assumption, we have $f = \sum_{n=1}^{\infty}(f, e_n)e_n$ and, by Bessel's Inequality, we know that $\sum_{n=1}^{\infty} |(f, e_n)|^2 < \infty$. Next, choose an integer m such that $\left[\sum_{k=m}^{\infty} |(f, e_k)|^2\right]^{\frac{1}{2}} < \epsilon$, and then let $h = \sum_{k=1}^{m}(f, e_k)e_k$. Then, h is in the linear span of the sequence $\{e_n\}$, and moreover

$$\begin{aligned}\|g - h\| &= \Big\|g - \sum_{k=1}^{m}(f, e_k)e_k\Big\| \leq \|g - f\| + \Big\|\sum_{k=m+1}^{\infty}(f, e_k)e_k\Big\| \\ &\leq \|g - f\| + \Big[\sum_{k=m+1}^{\infty} |(f, e_k)|^2\Big]^{\frac{1}{2}} < \epsilon + \epsilon = 2\epsilon.\end{aligned}$$

Therefore, the linear span of $\{e_n\}$ is dense and hence, $\{e_n\}$ is a complete orthonormal set, i.e., it is an orthonormal basis.

Problem 34.11. *Let $\{\phi_n\}$ be an orthonormal sequence of vectors in the Hilbert space $L_2[0, 2\pi]$. Suppose that for each continuous function f in $L_2[0, 2\pi]$ we have $\|f\|^2 = \sum_{n=1}^{\infty} |(f, \phi_n)|^2$. Show that $\{\phi_n\}$ is an orthonormal basis.*

Solution. It suffices to show that the linear span of the set $\{\phi_n\}$ is dense. Let $\epsilon > 0$ and let $g \in L_2[0, 2\pi]$. Since the continuous functions are dense in $L_2[0, 2\pi]$, there exists a continuous function $f \in L_2[0, 2\pi]$ with $\|f - g\| < \epsilon$.

Now, by our hypothesis, we have $\|f\|^2 = \sum_{n=1}^{\infty} |(f, \phi_n)|^2$. Choose an integer m such that $\left[\sum_{k=m}^{\infty} |(f, \phi_k)|^2\right]^{\frac{1}{2}} < \epsilon$, and note that

$$\Big\|g - \sum_{k=1}^{m}(f, \phi_k)\phi_k\Big\| \leq \|g - f\| + \Big\|f - \sum_{k=1}^{m}(f, \phi_k)\phi_k\Big\|.$$

Using once more our hypothesis, we see that

$$\begin{aligned}\Big\|g - \sum_{k=1}^{m}(f, \phi_k)\phi_k\Big\| &\leq \|g - f\| + \Big[\sum_{k=1}^{\infty}\Big|\Big(f - \sum_{i=1}^{m}(f, \phi_i)\phi_i, \phi_k\Big)\Big|^2\Big]^{\frac{1}{2}} \\ &\leq \|g - f\| + \Big[\sum_{k=m+1}^{\infty} |(f, \phi_k)|^2\Big]^{\frac{1}{2}} < \epsilon + \epsilon = 2\epsilon.\end{aligned}$$

Therefore, the linear span of $\{\phi_n\}$ is dense and hence, $\{\phi_n\}$ is an orthonormal basis.

Problem 34.12. *Let $\{\phi_1, \phi_2, \ldots\}$ be an orthonormal basis of the Hilbert space $L_2(\mu)$, where μ is a finite measure. Fix a function $f \in L_2(\mu)$ and let $\{\alpha_1, \alpha_2, \ldots\}$ be its sequence of Fourier coefficients relative to $\{\phi_n\}$, i.e., $\alpha_n = \int f \overline{\phi_n}\, d\mu$. Show that (although the series $\sum_{n=1}^{\infty} \alpha_n \phi_n$ need not converge pointwise almost everywhere to f) the Fourier series $\sum_{n=1}^{\infty} \alpha_n \phi_n$ can be integrated term-by-term in the sense that for every measurable set E we have*

$$\int_E f\, d\mu = \sum_{n=1}^{\infty} \alpha_n \int_E \phi_n\, d\mu.$$

Solution. Let $s_n = \sum_{k=1}^{n} \alpha_n \phi_n$, and note that $\|f - s_n\| \to 0$. Now, using the Cauchy–Schwarz inequality, we see that

$$\begin{aligned}\Big|\int_E f\, d\mu - \int_E s_n\, d\mu\Big|^2 &= \Big|\int_E (f - s_n)\, d\mu\Big|^2 \le \Big(\int_E |f - s_n|\, d\mu\Big)^2 \\ &\le \int_E |f - s_n|^2\, d\mu \cdot \int_E 1^2\, d\mu \\ &\le \|f - s_n\|^2 \mu^*(E) \longrightarrow 0.\end{aligned}$$

Hence, $\int_E f\, d\mu = \sum_{n=1}^{\infty} \alpha_n \int_E \phi_n\, d\mu$.

Problem 34.13. *Establish the following "perturbation" property of orthonormal bases. If $\{e_i\}_{i\in I}$ is an orthonormal basis and $\{f_i\}_{i\in I}$ is another orthonormal family satisfying*

$$\sum_{i\in I} \|e_i - f_i\|^2 < \infty,$$

then $\{f_i\}_{i\in I}$ is also an orthonormal basis.

Solution. Let $\{e_i\}_{i\in I}$ be an orthonormal basis in a Hilbert space H, and let $\{f_i\}_{i\in I}$ be another orthonormal family satisfying $\sum_{i\in I} \|e_i - f_i\|^2 < \infty$. To establish that the orthonormal family $\{f_i\}_{i\in I}$ is an orthonormal basis, it suffices to show that if a vector u satisfies $u \perp f_i$ for each $i \in I$, then $u = 0$. So, fix a vector $u \in H$ such that $u \perp f_i$ for all $i \in I$.

From $\sum_{i\in I} \|e_i - f_i\|^2 < \infty$, we know that the set $J = \{i \in I\colon f_i \neq e_i\}$ is at-most countable. We distinguish two cases.

CASE I: *J is finite, say $J = \{k_1, k_2, \ldots, k_\ell\}$.*

Let $M = \{y \in H\colon y \perp f_i \text{ for all } i \notin J\}$. Then, M is a closed vector subspace of H satisfying $\{f_{k_1}, \ldots, f_{k_\ell}\} \subseteq M$ and $\{e_{k_1}, \ldots, e_{k_\ell}\} \subseteq M$. Moreover, we claim that $\{e_{k_1}, \ldots, e_{k_\ell}\}$ must be an orthonormal basis for M. Indeed, if $x \in M$ satisfies

$z \perp e_{k_r}$ for $r = 1, \ldots, \ell$, then (in view $x \perp f_i = e_i$ for each $i \notin J$) we have $x \perp e_i$ for each $i \in I$. Since $\{e_i\}_{i \in I}$ is an orthonormal basis of H, it follows that $x = 0$. Thus, $\{e_{k_1}, \ldots, e_{k_\ell}\}$ is (as being an orthonormal basis) also a Hamel basis for M, and so M is ℓ-dimensional. This implies that $\{f_{k_1}, \ldots, f_{k_\ell}\}$ is also a Hamel basis. The latter implies that every e_{k_r} is a linear combination of the vectors $f_{k_1}, \ldots, f_{k_\ell}$. Consequently, $u \perp e_{k_r}$ for each $r = 1, \ldots, \ell$, and hence $u \perp e_i$ for all $i \in I$. This implies $u = 0$, and thus, in this case, $\{f_i\}_{i \in I}$ is an orthonormal basis.

CASE II: *J is countable, say $J = \{k_1, k_2, k_3, \ldots, \}$.*

In this case, choose a natural number ℓ such that

$$\sum_{j=\ell+1}^{\infty} \|e_{k_j} - f_{k_j}\|^2 = \delta < 1,$$

and let $J_1 = \{k_1, k_2, \ldots, k_\ell\}$. Next, define the vectors

$$g_r = e_{k_r} - \sum_{j=\ell+1}^{\infty} (e_{k_r}, f_{k_j}) f_{k_j}, \quad r = 1, 2, \ldots, \ell.$$

We claim the following:

- *If a vector $x \in H$ satisfies $x \perp g_r$ for $r = 1, 2, \ldots, \ell$ and $x \perp f_j$ for $j \notin J_1$, then $x = 0$.*

To see this, assume that vector $x \in H$ is orthogonal to g_r for $r = 1, 2, \ldots, \ell$ and to each f_j for $j \notin J_1$. Then, for $j \notin J_1$, we have $(x, e_j) = (x, e_j - f_j)$ and for each $1 \le r \le \ell$, we have

$$(x, e_{k_r}) = (x, g_r) + \sum_{j=\ell+1}^{\infty} (e_{k_r}, f_{k_j})(x, f_{k_j}) = 0.$$

Now, from Parseval's Identity, we have

$$\begin{aligned} \|x\|^2 &= \sum_{i \in I} |(x, e_i)|^2 = \sum_{j=\ell+1}^{\infty} |(x, e_{k_j} - f_{k_j})|^2 \\ &\le \Big[\sum_{j=\ell+1}^{\infty} \|e_{k_j} - f_{k_j}\|^2 \Big] \|x\|^2 = \delta \|x\|^2. \end{aligned}$$

This implies $0 \le (1 - \delta)\|x\|^2 \le 0$, or $x = 0$, as claimed.

Next, we consider the closed vector subspace

$$M = \{y \in H\colon\ y \perp f_i \ \text{ for all } \ i \notin J_1\}.$$

Clearly, $\{g_1, g_2, \ldots, g_\ell\} \subseteq M$. Moreover, if some vector $x \in M$ is orthogonal to $g_1, g_2, \ldots, g_\ell$, then by property ($\bullet$) we have $x = 0$. This means that the vector space generated by $g_1, g_2, \ldots, g_\ell$ coincides with M. Since the orthogonal vectors $f_{k_1}, f_{k_2}, \ldots, f_{k_\ell}$ belong to M, M is ℓ-dimensional and so $\{f_{k_1}, f_{k_2}, \ldots, f_{k_\ell}\}$ is a Hamel basis of M. In particular, for each $1 \le r \le \ell$ the vector g_r is a linear combination of the vectors $f_{k_1}, f_{k_2}, \ldots, f_{k_\ell}$. This implies $u \perp g_r$ for each $1 \le r \le \ell$ and $u \perp f_j$ for $j \notin J_1$. Using ($\bullet$) once more, we conclude that $u = 0$. Therefore, the orthonormal family $\{f_i\}_{i \in I}$ is an orthonormal basis.

35. FOURIER ANALYSIS

Problem 35.1. *Show that* $\sin^n x$ *is a linear combination of*

$$\{1, \sin x, \cos x, \sin 2x, \cos 2x, \sin 3x, \cos 3x, \ldots, \sin nx, \cos nx\}.$$

Furthermore, show that the coefficients of the cosine terms are zero when n is an odd integer, and the coefficients of the sine terms are zero when n is an even integer.

Solution. Observe that

$$\sin x = \frac{e^{\imath x} - e^{-\imath x}}{2\imath}.$$

Then, using the binomial theorem, we get

$$\begin{aligned}
\sin^n x &= \Big(\frac{e^{\imath x} - e^{-\imath x}}{2\imath}\Big)^n = \frac{1}{(2\imath)^n}\sum_{k=0}^{n}\Big[\frac{n!(-1)^k}{k!(n-k)!}e^{\imath(n-k)x}e^{-\imath kx}\Big]\\
&= \frac{1}{(2\imath)^n}\sum_{k=0}^{n}\Big[\frac{n!(-1)^k}{k!(n-k)!}e^{\imath(n-2k)x}\Big]\\
&= \frac{1}{(2\imath)^n}\sum_{k=0}^{n}\Big[\frac{n!(-1)^k}{k!(n-k)!}\big(\cos(n-2k)x + \imath\sin(n-2k)x\big)\Big]\\
&= \frac{1}{(2\imath)^n}\sum_{k=0}^{n}\Big[\frac{n!(-1)^k}{k!(n-k)!}\cos(n-2k)x\Big] + \frac{\imath}{(2\imath)^n}\sum_{k=0}^{n}\Big[\frac{n!(-1)^k}{k!(n-k)!}\sin(n-2k)x\Big].
\end{aligned}$$

Now, observe that if n is odd, then $\imath^n = \pm\imath$ and if n is even, then $\imath^n = \pm 1$. Since

$\sin^n x$ is equal to the real part of the preceding expression, we have the following two cases:

$$\sin^n x = \frac{1}{(2\imath)^n}\sum_{k=0}^{n}\left[\frac{n!(-1)^k}{k!(n-k)!}\cos(n-2k)x\right] \quad \text{for } n \text{ even, and}$$

$$\sin^n x = \frac{\imath}{(2\imath)^n}\sum_{k=0}^{n}\left[\frac{n!(-1)^k}{k!(n-k)!}\sin(n-2k)x\right] \quad \text{for } n \text{ odd.}$$

Problem 35.2. *Show that the Dirichlet kernel D_n and the Fejér kernel K_n satisfy*

$$\frac{1}{\pi}\int_{-\pi}^{\pi} D_n(t)\,dt = \frac{1}{\pi}\int_{-\pi}^{\pi} K_n(t)\,dt = 1.$$

Solution. The Dirichlet kernel is given by

$$D_n(t) = \frac{1}{2} + \sum_{k=1}^{n}\cos kt.$$

Integrating gives

$$\frac{1}{\pi}\int_{-\pi}^{\pi} D_n(t)\,dt = 1 + \frac{1}{\pi}\sum_{k=1}^{n}\int_{-\pi}^{\pi}\cos kt\,dt = 1.$$

Likewise, the Fejér kernel is defined by

$$K_n(t) = \frac{1}{n+1}\sum_{k=0}^{n} D_k(t).$$

So, integrating and using the previous result on the Dirichlet kernel, we get

$$\frac{1}{\pi}\int_{-\pi}^{\pi} K_n(t)\,dt = \frac{1}{n+1}\sum_{k=0}^{n}\frac{1}{\pi}\int_{-\pi}^{\pi} D_n(t)\,dt = \frac{1}{n+1}\sum_{k=0}^{n} 1 = 1.$$

Problem 35.3. *Let X denote the Banach space of all continuous periodic real-valued functions defined on $[0,2\pi]$. Fix some $x \in [0,2\pi]$ and define the linear*

functional $S_n: X \to \mathbb{R}$ by the formula

$$S_n(f) = \frac{1}{\pi}\int_0^{2\pi} f(t)D_n(x-t)\,dt.$$

Show that the norm of the linear functional S_n satisfies

$$\|S_n\| = \frac{1}{\pi}\int_0^{2\pi} |D_n(x-t)|\,dt.$$

Solution. The norm of S_n is defined by

$$\|S_n\| = \sup_{\|f\|_\infty \le 1} \left| \frac{1}{\pi}\int_0^{2\pi} f(t)D_n(x-t)\,dt \right|.$$

Since $\|f\|_\infty \le 1$ implies $|f(t)D(x-t)| \le |D(x-t)|$ for each t, we see that

$$\|S_n\| \le \frac{1}{\pi}\int_0^{2\pi} |D_n(x-t)|\,dt.$$

Next, we shall establish the reverse inequality. Since the Dirichlet kernel D_n has period 2π, it follows that the continuous function

$$f(\epsilon, t) = \frac{D_n(x-t)}{|D_n(x-t)| + \epsilon}$$

also has period 2π with respect to t, and clearly $|f(\epsilon, t)| \le 1$ for each t and all $\epsilon > 0$. This implies

$$\|S_n\| \ge S_n(f) = \frac{1}{\pi}\int_0^{2\pi} \frac{|D_n(x-t)|^2}{|D_n(x-t)| + \epsilon}\,dt.$$

Taking into account Theorem 24.4 and letting $\epsilon \to 0^+$ yields

$$\|S_n\| \ge \frac{1}{\pi}\int_0^{2\pi} |D_n(x-t)|\,dt.$$

Therefore, $\|S_n\| = \frac{1}{\pi}\int_0^{2\pi} |D_n(x-t)|\,dt$ holds true.

Problem 35.4. *Show that the sequence of functions*

$$\{(\tfrac{1}{\pi})^{\frac{1}{2}},\ (\tfrac{2}{\pi})^{\frac{1}{2}}\cos x,\ (\tfrac{2}{\pi})^{\frac{1}{2}}\cos 2x,\ (\tfrac{2}{\pi})^{\frac{1}{2}}\cos 3x\,,\ (\tfrac{2}{\pi})^{\frac{1}{2}}\cos 4x, \dots\}$$

is an orthonormal basis in $L_2[0,\pi]$. Also show that the preceding sequence is an orthogonal sequence of functions in $L_2[0,2\pi]$ which is not complete.

Solution. It is easy to verify that the functions

$$(\tfrac{1}{\pi})^{\frac{1}{2}},\ (\tfrac{2}{\pi})^{\frac{1}{2}}\cos x,\ (\tfrac{2}{\pi})^{\frac{1}{2}}\cos 2x,\ (\tfrac{2}{\pi})^{\frac{1}{2}}\cos 3x, \dots$$

are mutually orthogonal and of norm one in $L_2[0,\pi]$.

To show that the preceding orthonormal sequence is complete (i.e., that it is an orthonormal basis), we need to show that if $f \in L_2[0,\pi]$ is perpendicular to the functions $1, \cos x, \cos 2x, \cos 3x, \dots$, then $f = 0$. To this end, suppose that a function $f \in L_2[0,\pi]$ satisfies

$$\int_0^{\pi} f(x)\cos nx\,dx = 0$$

for all $n = 0, 1, 2, \dots$. Define the function $g\colon [0,2\pi] \to \mathbf{R}$ by

$$g(x) = \begin{cases} f(x) & \text{if } 0 \le x \le \pi \\ f(2\pi - x) & \text{if } \pi < x \le 2\pi. \end{cases}$$

Then, in $L_2[0,2\pi]$, for each n we have

$$\begin{aligned}(g, \cos nx) &= \int_0^{2\pi} g(x)\cos nx\,dx \\ &= \int_0^{\pi} f(x)\cos nx\,dx + \int_{\pi}^{2\pi} f(2\pi - x)\cos nx\,dx.\end{aligned}$$

The change of variable $t = 2\pi - x$ gives

$$(g, \cos nx) = \int_0^{\pi} f(t)\cos nt\,dt + \int_0^{\pi} f(t)\cos nt\,dt = 0.$$

Next, observe that

$$\begin{aligned}(g, \sin nx) &= \int_0^{2\pi} g(x)\sin nx\,dx = \int_0^{\pi} f(x)\sin nx\,dx + \int_{\pi}^{2\pi} f(2\pi - x)\sin nx\,dx \\ &= \int_0^{\pi} f(t)\sin nt\,dt - \int_0^{\pi} f(t)\sin nt\,dt = 0.\end{aligned}$$

The preceding show that g is perpendicular to every vector of a complete orthogonal sequence of $L_2[0, 2\pi]$. Therefore, $g = 0$ and hence, $f = 0$. Thus, the sequence

$$\left(\tfrac{1}{\pi}\right)^{\frac{1}{2}},\ \left(\tfrac{2}{\pi}\right)^{\frac{1}{2}}\cos x,\ \left(\tfrac{2}{\pi}\right)^{\frac{1}{2}}\cos 2x,\ \left(\tfrac{2}{\pi}\right)^{\frac{1}{2}}\cos 3x, \ldots$$

is an orthonormal basis of $L_2[0, \pi]$.

To see that the orthogonal set

$$\left(\tfrac{1}{\pi}\right)^{\frac{1}{2}},\ \left(\tfrac{2}{\pi}\right)^{\frac{1}{2}}\cos x,\ \left(\tfrac{2}{\pi}\right)^{\frac{1}{2}}\cos 2x,\ \left(\tfrac{2}{\pi}\right)^{\frac{1}{2}}\cos 3x, \ldots$$

is not complete in $L_2[0, 2\pi]$ notice that the nonzero function $\sin x$ is perpendicular to each of these functions in $L_2[0, 2\pi]$.

Problem 35.5. *Show that the sequence of functions*

$$\left\{\left(\tfrac{2}{\pi}\right)^{\frac{1}{2}}\sin x,\ \left(\tfrac{2}{\pi}\right)^{\frac{1}{2}}\sin 2x,\ \left(\tfrac{2}{\pi}\right)^{\frac{1}{2}}\sin 3x,\ \left(\tfrac{2}{\pi}\right)^{\frac{1}{2}}\sin 4x,\ \ldots\right\}$$

is an orthonormal basis of $L_2[0, \pi]$. Also prove that this set of functions is an orthogonal set of functions in $L_2[0, 2\pi]$ which is not complete.

Solution. It is easy to verify that the collection of functions

$$\left\{\left(\tfrac{2}{\pi}\right)^{\frac{1}{2}}\sin x,\ \left(\tfrac{2}{\pi}\right)^{\frac{1}{2}}\sin 2x,\ \left(\tfrac{2}{\pi}\right)^{\frac{1}{2}}\sin 3x,\ \left(\tfrac{2}{\pi}\right)^{\frac{1}{2}}\sin 4x,\ \ldots\right\}$$

is an orthonormal set of functions of $L_2[0, \pi]$.

Thus, in order to show that it is an orthonormal basis, we need to show that if a function $f \in L_2[0, \pi]$ is perpendicular to each function of the preceding set in $L_2[0, \pi]$, then $f = 0$. To this end, assume that a function $f \in L_2[0, \pi]$ satisfies

$$\int_0^{\pi} f(x) \sin nx \, dx = 0$$

for all $n = 1, 2, 3, \ldots$. Define the function $g\colon [0, 2\pi] \to \mathbf{R}$ by

$$g(x) = \begin{cases} f(x) & \text{if } 0 \le x \le \pi \\ -f(2\pi - x) & \text{if } \pi < x \le 2\pi\,. \end{cases}$$

Then, in $L_2[0, 2\pi]$, for each n we have

$$
\begin{aligned}
(g, \sin nx) &= \int_0^{2\pi} g(x) \sin nx \, dx \\
&= \int_0^{\pi} f(x) \sin nx \, dx - \int_{\pi}^{2\pi} f(2\pi - x) \sin nx \, dx \\
&= \int_0^{\pi} f(t) \sin nt \, dt + \int_0^{\pi} f(t) \sin nt \, dt = 0 + 0 = 0.
\end{aligned}
$$

Next, observe that

$$
\begin{aligned}
(g, \cos nx) &= \int_0^{2\pi} g(x) \cos nx \, dx \\
&= \int_0^{\pi} f(x) \cos nx \, dx - \int_{\pi}^{2\pi} f(2\pi - x) \cos nx \, dx \\
&= \int_0^{\pi} f(t) \cos nt \, dt - \int_0^{\pi} f(t) \cos nt \, dt = 0.
\end{aligned}
$$

Therefore, g is perpendicular to every vector of a complete orthogonal set of functions in $L_2[0, 2\pi]$. Therefore, $g = 0$ and hence, $f = 0$. Thus, the orthonormal set of functions

$$\left\{ \left(\tfrac{2}{\pi}\right)^{\frac{1}{2}} \sin x\,,\ \left(\tfrac{2}{\pi}\right)^{\frac{1}{2}} \sin 2x,\ \left(\tfrac{2}{\pi}\right)^{\frac{1}{2}} \sin 3x\,,\ \left(\tfrac{2}{\pi}\right)^{\frac{1}{2}} \sin 4x,\ \ldots \right\}$$

is an orthonormal basis of $L_2[0, \pi]$.

To see that the orthogonal set of functions

$$\left\{ \left(\tfrac{2}{\pi}\right)^{\frac{1}{2}} \sin x\,,\ \left(\tfrac{2}{\pi}\right)^{\frac{1}{2}} \sin 2x,\ \left(\tfrac{2}{\pi}\right)^{\frac{1}{2}} \sin 3x\,,\ \left(\tfrac{2}{\pi}\right)^{\frac{1}{2}} \sin 4x,\ \ldots \right\}$$

is not complete in $L_2[0, 2\pi]$, observe that $\cos x$ is perpendicular to each of these functions.

Problem 35.6. *The original Weierstrass approximation theorem showed that every continuous function of period 2π can be uniformly approximated by trigonometric polynomials. Establish this result.*

Solution. Weierstrass originally gave a direct proof, however, the result can be derived directly from Fejér's Theorem 35.8. Let f be a continuous function of period 2π defined on the entire real line. Let $\epsilon > 0$ be fixed. Let $\{s_n\}$ be the sequence of partial sums of the Fourier series of f and let $\{\sigma_n\}$ be the sequence arithmetic means.

From Theorem 35.8 we know that the sequence $\{\sigma_n\}$ converges uniformly to f on $[0, 2\pi]$. Now, notice that each σ_n is a trigonometric polynomial, and the claim is established.

Problem 35.7. *Find the Fourier coefficients of the function*

$$f(x) = \begin{cases} 1 & \text{if } 0 \le x < \frac{\pi}{2} \\ 0 & \text{if } \frac{\pi}{2} \le x < 2\pi. \end{cases}$$

Solution. The Fourier coefficients are given by the formulas

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} dx = \frac{1}{2}, \\ a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} \cos nx\,dx = \frac{1}{n\pi}\sin\left(n\tfrac{\pi}{2}\right), \text{ and} \\ b_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} \sin nx\,dx = -\frac{1}{n\pi}\left[\cos\left(n\tfrac{\pi}{2}\right) - 1\right]. \end{aligned}$$

Simplifying yields

$$\begin{aligned} a_0 &= \tfrac{1}{2}, \\ a_n &= \begin{cases} 0 & \text{if } n = 2, 4, 6, \ldots \\ 1/n\pi & \text{if } n = 1, 5, 9, \ldots \\ -1/n\pi & \text{if } n = 3, 7, 11, \ldots, \end{cases} \\ b_n &= \begin{cases} 1/n\pi & \text{if } n = 1, 3, 5, \ldots \\ 2/n\pi & \text{if } n = 2, 6, 10, \ldots \\ 0 & \text{if } n = 4, 8, 12, \ldots. \end{cases} \end{aligned}$$

Problem 35.8. *Find the Fourier series of the function*

$$f(x) = \begin{cases} \sin x & \text{if } 0 \le x < \pi \\ -\sin x & \text{if } \pi \le x < 2\pi. \end{cases}$$

Solution. The function f is continuous, even, and periodic. Its Fourier coefficients are given by

$$\begin{aligned} a_0 &= \tfrac{2}{\pi}\int_0^{\pi} \sin x\,dx = \tfrac{4}{\pi}, \\ a_n &= \tfrac{2}{\pi}\int_0^{\pi} \sin x \cos nx\,dx = \begin{cases} -\frac{4}{\pi(n^2-1)} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd, and} \end{cases} \\ b_n &= 0. \end{aligned}$$

So, the Fourier series of f is given by $\frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos 2nx}{4n^2-1}$. Since this series converges at every x and the periodic function f is continuous everywhere, it follows from Corollary 35.9 that the series converges to $f(x)$ for each x. That is, we have

$$f(x) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos 2nx}{4n^2-1}$$

for each real number x.

Problem 35.9. *Show that for each* $0 < x < 2\pi$ *we have*

$$x = \pi - 2\sum_{n=1}^{\infty}\frac{\sin nx}{n}.$$

Solution. We consider the periodic function $f\colon [0, 2\pi] \to \mathbf{R}$ defined by

$$f(x) = \begin{cases} x & \text{if } 0 \le x < 2\pi \\ 0 & \text{if } x = 2\pi\,. \end{cases}$$

Computing the Fourier coefficients of f, we obtain

$$a_0 = \tfrac{1}{\pi}\int_0^{2\pi} x\,dx = \tfrac{x^2}{2\pi}\Big|_0^{2\pi} = 2\pi,$$

$$\begin{aligned} a_n &= \tfrac{1}{\pi}\int_0^{2\pi} x\cos nx\,dx = \tfrac{1}{n\pi}\int_0^{2\pi} x\,d(\sin nx) \\ &= \tfrac{1}{n\pi}\Big[x\sin nx\Big|_0^{2\pi} - \int_0^{2\pi}\sin nx\,dx\Big] = 0, \text{ and} \end{aligned}$$

$$\begin{aligned} b_n &= \tfrac{1}{\pi}\int_0^{2\pi} x\sin nx\,dx = -\tfrac{1}{n\pi}\int_0^{2\pi} x\,d(\cos nx) \\ &= -\tfrac{1}{n\pi}\Big[x\cos nx\Big|_0^{2\pi} - \int_0^{2\pi}\cos nx\,dx\Big] = -\tfrac{1}{n\pi}\Big[2\pi - \tfrac{1}{n}\sin nx\Big|_0^{2\pi}\Big] = -\tfrac{2}{n}. \end{aligned}$$

So, the Fourier series of the function f is $\pi - 2\sum_{n=1}^{\infty}\frac{\sin nx}{n}$. Given that the function f is continuous at every $0 < x < 2\pi$ and that the preceding Fourier series converges for each $0 < x < 2\pi$ (see Example 9.7), it follows from Corollary 35.9 that

$$x = \pi - 2\sum_{n=1}^{\infty}\frac{\sin nx}{n}$$

holds for each $0 < x < 2\pi$.

Problem 35.10. *Show that*

$$\frac{x^2}{2} = \pi x - \frac{\pi^2}{3} + 2\sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$$

holds for all $0 \le x \le 2\pi$. *Letting* $x = 0$ *we obtain the formula* $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

Solution. Consider the periodic function $f\colon [0, 2\pi] \to \mathbf{R}$ defined by $f(x) = \frac{x^2}{2} - \pi x$. Computing its Fourier coefficients, we get

$$a_0 = \tfrac{1}{\pi} \int_0^{2\pi} \left(\tfrac{x^2}{2} - \pi x\right) dx = \tfrac{1}{\pi}\left(\tfrac{x^3}{6} - \tfrac{\pi x^2}{2}\right)\Big|_0^{2\pi} = -\tfrac{2\pi^2}{3},$$

$$a_n = \tfrac{1}{\pi} \int_0^{2\pi} \left(\tfrac{x^2}{2} - \pi x\right) \cos nx \, dx = \tfrac{2}{n^2}, \text{ and}$$

$$b_n = \tfrac{1}{\pi} \int_0^{2\pi} \left(\tfrac{x^2}{2} - \pi x\right) \sin nx \, dx = 0.$$

Therefore, the Fourier series of the function f is $\frac{\pi^2}{3} + 2\sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$. Since this series converges for each x and f is a continuous function, it follows from Corollary 35.9 that

$$\frac{x^2}{2} - \pi x = -\frac{\pi^2}{3} + 2\sum_{n=1}^{\infty} \frac{\cos nx}{n^2},$$

and the desired identity follows.

Problem 35.11. *Show that*

$$x^2 = \tfrac{4}{3}\pi^2 + 4\sum_{n=1}^{\infty}\left(\frac{\cos nx}{n^2} - \frac{\pi \sin nx}{n}\right)$$

holds for each $0 < x < 2\pi$.

Solution. Consider the periodic function $f\colon [0, 2\pi] \to \mathbf{R}$ defined by

$$f(x) = \begin{cases} x^2 & \text{if } 0 \le x < 2\pi \\ 0 & \text{if } x = 2\pi\,. \end{cases}$$

Computing the Fourier coefficients of f, we obtain

$$a_0 = \tfrac{1}{\pi} \int_0^{2\pi} x^2 \, dx = \tfrac{x^3}{3\pi}\Big|_0^{2\pi} = \tfrac{8}{3}\pi^2,$$

$$a_n = \tfrac{1}{\pi}\int_0^{2\pi} x^2 \cos nx\, dx = \tfrac{4}{n^2}, \text{ and}$$

$$b_n = \tfrac{1}{\pi}\int_0^{2\pi} x^2 \sin nx\, dx = -\tfrac{4\pi}{n}.$$

Therefore, the Fourier series of the function f is $\frac{4}{3}\pi^2 + 4\sum_{n=1}^{\infty}\left(\frac{\cos nx}{n^2} - \frac{\pi \sin nx}{n}\right)$. Since this series converges for each x (see Example 9.7) and f is continuous at each $0 < x < 2\pi$, it follows from Corollary 35.9 that

$$x^2 = \tfrac{4}{3}\pi^2 + 4\sum_{n=1}^{\infty}\left(\frac{\cos nx}{n^2} - \frac{\pi \sin nx}{n}\right),$$

for each $0 < x < 2\pi$.

Problem 35.12. *Consider the "integral" operator $T\colon L_2[0,\pi] \to L_2[0,\pi]$ defined by*

$$Tf(x) = \int_0^{\pi} K(x,t) f(t)\, dt,$$

where the kernel $K\colon [0,\pi]\times[0,\pi] \to \mathbb{R}$ is given by

$$K(x,t) = \sum_{n=1}^{\infty} \frac{[\sin(n+1)x]\sin nt}{n^2}.$$

Show that the norm of the operator T satisfies $\|T\| = \pi/2$.

Solution. By Problem 35.5, we know that the sequence of functions

$$\left\{\left(\tfrac{2}{\pi}\right)^{\frac{1}{2}} \sin nx\colon\ n = 1, 2, \ldots\right\}$$

is an orthonormal basis for $L_2[0,\pi]$. Also, as usual, the norm of the operator is given by

$$\|T\| = \sup\left\{\|T(f)\|\colon\ f \in L_2[0,\pi] \text{ and } \|f\| = 1\right\}.$$

Now, fix a function $f \in L_2[0,\pi]$ with $\|f\| = 1$, and write

$$f = \sum_{n=1}^{\infty} c_n \left(\tfrac{2}{\pi}\right)^{\frac{1}{2}} \sin nx$$

in its Fourier expansion relative to the above basis. By Parseval's Identity, we have

$$\|f\|^2 = \sum_{n=1}^{\infty} |c_n|^2.$$

Next, notice that the operator satisfies

$$\begin{aligned}
Tf(x) &= \int_0^{\pi} \Big[\sum_{n=1}^{\infty} \tfrac{\sin(n+1)x \, \sin nt}{n^2} \Big] \Big[\sum_{m=1}^{\infty} c_m \big(\tfrac{2}{\pi}\big)^{\frac{1}{2}} \sin mt \Big] dt \\
&= \sum_{n=1}^{\infty} \Big[\tfrac{\sin(n+1)x}{n^2} \Big(\int_0^{\pi} \sin nt \sum_{m=1}^{\infty} c_m \big(\tfrac{2}{\pi}\big)^{\frac{1}{2}} \sin mt \, dt \Big) \Big] \\
&= \tfrac{\pi}{2} \sum_{n=1}^{\infty} \Big[\tfrac{c_n}{n^2} \big(\tfrac{2}{\pi}\big)^{\frac{1}{2}} \sin(n+1)x \Big].
\end{aligned}$$

Now, notice that the latter expression is the Fourier expansion of $T(f)$ with respect to the orthonormal basis described at the beginning of the solution. Thus, by Parseval's Identity, we have

$$\|T(f)\|^2 = \tfrac{\pi^2}{4} \sum_{n=1}^{\infty} \tfrac{|c_n|^2}{n^4} \le \tfrac{\pi^2}{4} \sum_{n=1}^{\infty} |c_n|^2 = \tfrac{\pi^2}{4} \|f\|^2,$$

from which it follows that $\|T\| \le \pi/2$ holds.

Finally, if $f_0(x) = (\frac{2}{\pi})^{\frac{1}{2}} \sin x$, then $\|f_0\| = 1$ and by Parseval's Identity we have $c_1 = 1$ and $c_n = 0$ for $n \ne 1$, and so $\|T(f_0)\|^2 = \frac{\pi^2}{4}$. Therefore, $\|T\| \ge \pi/2$, and hence, $\|T\| = \pi/2$.

CHAPTER 7

SPECIAL TOPICS IN INTEGRATION

36. SIGNED MEASURES

Problem 36.1. *Give an example of a signed measure and two Hahn decompositions (A, B) and (A_1, B_1) of X with respect to the signed measure such that $A \neq A_1$ and $B \neq B_1$.*

Solution. Let $X = \mathbb{R}$ and let Σ be the σ-algebra of all Lebesgue measurable sets. Consider the measures $\mu_1, \mu_2 \in M(\Sigma)$ defined by $\mu_1(E) = \lambda(E \cap [0, 1])$ and $\mu_2(E) = \lambda(E \cap [1, 2])$ for each $E \in \Sigma$ (where λ denotes the Lebesgue measure on $\mathbb{R}$). Now, consider the signed measure $\mu = \mu_1 - \mu_2$, and note that $\big((-\infty, 1), [1, \infty)\big)$ and $\big([0, 1), (-\infty, 0) \cup [1, \infty)\big)$ are two Hahn decompositions of X with respect to the signed measure μ.

Problem 36.2. *If μ is a signed measure, then show that $\mu^+ \wedge \mu^- = 0$.*

Solution. Let (A, B) be a Hahn decomposition of X with respect to μ. If $E \in \Sigma$, then note that

$$\begin{aligned} 0 \le \mu^+ \wedge \mu^-(E) &= \mu^+ \wedge \mu^-(E \cap B) + \mu^+ \wedge \mu^-(E \cap A) \\ &\le \mu^+(E \cap B) + \mu^-(E \cap A) \\ &= \mu(E \cap B \cap A) - \mu(E \cap A \cap B) = 0. \end{aligned}$$

Problem 36.3. *If μ is a signed measure, then show that for each $A \in \Sigma$ we have*

$$|\mu|(A) = \sup\left\{ \sum_{n=1}^{\infty} |\mu(A_n)| \colon \{A_n\} \text{ is a disjoint sequence of } \Sigma \text{ with } \bigcup_{n=1}^{\infty} A_n = A \right\}.$$

Solution. Fix $A \in \Sigma$. From Theorem 36.9, we know that

$$|\mu|(A) = \sup\Big\{\sum_{n=1}^{k}|\mu(A_n)|\colon \{A_1, \ldots, A_k\} \subseteq \Sigma \text{ is disjoint and } \bigcup_{n=1}^{k} A_n \subseteq A\Big\}.$$

Also, let

$$s = \sup\Big\{\sum_{n=1}^{\infty}|\mu(A_n)|\colon \{A_n\} \subseteq \Sigma \text{ is disjoint and } A = \bigcup_{n=1}^{\infty} A_n\Big\}.$$

Now, let $\{A_n\}$ be a pairwise disjoint sequence of Σ such that $\bigcup_{n=1}^{\infty} A_n = A$. Clearly, $\sum_{n=1}^{k} |\mu(A_n)| \leq |\mu|(A)$ holds for each k, and so

$$\sum_{n=1}^{\infty} |\mu(A_n)| = \lim_{k\to\infty} \sum_{n=1}^{k} |\mu(A_n)| \leq |\mu|(A).$$

Therefore, $s \leq |\mu|(A)$. On the other hand, if $\{A_1, \ldots, A_k\}$ is a finite pairwise disjoint collection of Σ satisfying $\bigcup_{n=1}^{k} A_n \subseteq A$, then

$$A = A_1 \cup \cdots \cup A_k \cup \Big(A \setminus \bigcup_{n=1}^{k} A_k\Big) \cup \emptyset \cup \emptyset \cdots,$$

and so

$$\sum_{n=1}^{k}|\mu(A_n)| \leq \sum_{n=1}^{k}|\mu(A_n)| + \Big|\mu\Big(A \setminus \bigcup_{n=1}^{k} A_n\Big)\Big| + \mu(\emptyset) + \mu(\emptyset) + \cdots \leq s.$$

Consequently, $|\mu|(A) \leq s$ also holds. Thus, $|\mu|(A) = s$, as claimed.

Problem 36.4. *Verify that if μ and ν are two finite signed measures, then the least upper bound $\mu \vee \nu$ and the greatest lower bound $\mu \wedge \nu$ holds in $M(\Sigma)$ are given by*

$$\mu \vee \nu(A) = \sup\{\mu(B) + \nu(A \setminus B)\colon B \in \Sigma \text{ and } B \subseteq A\}, \text{ and}$$
$$\mu \wedge \nu(A) = \inf\{\mu(B) + \nu(A \setminus B)\colon B \in \Sigma \text{ and } B \subseteq A\}$$

for each $A \in \Sigma$.

Solution. The proof parallels the one of Theorem 36.1. We shall verify that if $\mu, \nu \in M(\Sigma)$, then the formula

$$\omega(A) = \inf\{\mu(B) + \nu(A \setminus B)\colon\ B \in \Sigma \text{ and } B \subseteq A\}, \quad A \in \Sigma,$$

defines a finite signed measure (i.e., $\omega \in M(\Sigma)$), and that ω is the greatest lower bound of μ and ν in $M(\Sigma)$.

Since μ and ν are both bounded from below (and also both bounded from above), it follows that $\omega(A) \in \mathbb{R}$ for each $A \in \Sigma$. Clearly, $\omega(\emptyset) = 0$ holds. Next, we shall establish that ω is σ-additive. To this end, let $\{A_n\}$ be a pairwise disjoint sequence of Σ and let $A = \bigcup_{n=1}^{\infty} A_n$. If $B \in \Sigma$ satisfies $B \subseteq A$, then

$$\begin{aligned}
\mu(B) + \nu(A \setminus B) &= \mu\Big(\bigcup_{n=1}^{\infty} A_n \cap B\Big) + \nu\Big(\bigcup_{n=1}^{\infty}(A_n \setminus A_n \cap B)\Big) \\
&= \sum_{n=1}^{\infty}[\mu(A_n \cap B) + \nu(A_n \setminus A_n \cap B)] \\
&\geq \sum_{n=1}^{\infty} \omega(A_n),
\end{aligned}$$

and so $\omega(A) \geq \sum_{n=1}^{\infty} \omega(A_n)$ holds. For the reverse inequality, let $\varepsilon > 0$. Then, for each n pick some $B_n \in \Sigma$ with $B_n \subseteq A_n$ and

$$\mu(B_n) + \nu(A_n \setminus B_n) < \omega(A_n) + \frac{\varepsilon}{2^n}.$$

Obviously, $\{B_n\}$ is a pairwise disjoint sequence of Σ. Put $B = \bigcup_{n=1}^{\infty} B_n \subseteq A$, and note that $\bigcup_{n=1}^{\infty}(A_n \setminus B_n) = A \setminus B$ holds. Moreover,

$$\begin{aligned}
\omega(A) &\leq \mu(B) + \nu(A \setminus B) = \mu\Big(\bigcup_{n=1}^{\infty} B_n\Big) + \nu\Big(\bigcup_{n=1}^{\infty}(A_n \setminus B_n)\Big) \\
&= \sum_{n=1}^{\infty}[\mu(B_n) + \nu(A_n \setminus B_n)] \\
&\leq \sum_{n=1}^{\infty}\big[\omega(A_n) + \tfrac{\varepsilon}{2^n}\big] = \sum_{n=1}^{\infty} \omega(A_n) + \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we infer that $\omega(A) \leq \sum_{n=1}^{\infty} \omega(A_n)$ also holds, and so $\omega \in M(\Sigma)$.

Finally, we shall establish that ω is the greatest lower bound of μ and ν in $M(\Sigma)$. Note first that ω is a lower bound for both μ and ν. Indeed, if $A \in \Sigma$, then (by letting $B = A$), we see that

$$\omega(A) \le \mu(A) + \nu(\emptyset) = \mu(A) \quad \text{and} \quad \omega(A) \le \mu(\emptyset) + \nu(A) = \nu(A).$$

On the other hand, if $\pi \in M(\Sigma)$ satisfies $\pi \le \mu$ and $\pi \le \nu$ and $A \in \Sigma$, then for each $B \in \Sigma$ with $B \subseteq A$ we have

$$\pi(A) = \pi(B) + \pi(A \setminus B) \le \mu(B) + \nu(A \setminus B),$$

from which it follows that $\pi(A) \le \omega(A)$, i.e., $\pi \le \omega$. This shows that $\omega = \mu \wedge \nu$ holds in $M(\Sigma)$.

Problem 36.5. *Let λ be the Lebesgue measure on the Lebesgue measurable subsets of $\mathbb{R}$. If μ is the Dirac measure, defined by $\mu(A) = 0$ if $0 \notin A$ and $\mu(A) = 1$ if $0 \in A$, describe $\lambda \vee \mu$ and $\lambda \wedge \mu$.*

Solution. If $A = \mathbb{R} \setminus \{0\}$, $B = \{0\}$, and E is an arbitrary Lebesgue measurable set, then

$$0 \le \lambda \wedge \mu(E) \le \lambda(E \cap B) + \mu(E \cap A) = 0$$

holds. That is, $\lambda \wedge \mu = 0$. Moreover, we have

$$\lambda \vee \mu = \lambda \vee \mu + \lambda \wedge \mu = \lambda + \mu.$$

Problem 36.6. *Show that the collection of all σ-finite measures forms a distributive lattice. That is, show that if μ, ν, and ω are three σ-finite measures, then*

$$(\mu \vee \nu) \wedge \omega = (\mu \wedge \omega) \vee (\nu \wedge \omega) \quad \textit{and} \quad (\mu \wedge \nu) \vee \omega = (\mu \vee \omega) \wedge (\nu \vee \omega).$$

Solution. We shall show first that every vector lattice satisfies the distributive law. To do this, we shall use the identity (a) of Problem 9.1.

Let x, y, and z be elements in a vector lattice. Since $x \vee y \ge x$, it follows that $(x \vee y) \wedge z \ge x \wedge z$, and similarly $(x \vee y) \wedge z \ge y \wedge z$. Thus,

$$(x \vee y) \wedge z \ge (x \wedge z) \vee (y \wedge z).$$

On the other hand, if $u = (x \wedge z) \vee (y \wedge z)$, then $u \ge x \wedge z = x + z - x \vee z$ holds.

Hence, $x \leq u-z+x\vee z \leq u-z+(x\vee y)\vee z$, and similarly $y \leq u-z+(x\vee y)\vee z$. It follows that $x \vee y \leq u - z + (x \vee y) \vee z$, and so

$$(x \wedge z) \vee (y \wedge z) = u \geq x \vee y + z - (x \vee y) \vee z = (x \vee y) \wedge z.$$

Therefore, $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$ holds. The other identity can be established in a similar manner.

Now, let $\{X_n\} \subseteq \Sigma$ satisfy $\mu(X_n) < \infty$, $\nu(X_n) < \infty$, $\omega(X_n) < \infty$ for all n, and $X_n \uparrow X$. If $\Sigma_n = \{A \cap X_n \colon A \in \Sigma\}$, then clearly μ, ν, and ω (restricted to Σ_n) belong to the vector lattice $M(\Sigma_n)$. Thus, if $E \in \Sigma$, then

$$(\mu \vee \nu) \wedge \omega(E \cap X_n) = (\mu \wedge \omega) \vee (\nu \wedge \omega)(E \cap X_n)$$

and

$$(\mu \wedge \nu) \vee \omega(E \cap X_n) = (\mu \vee \omega) \wedge (\nu \vee \omega)(E \cap X_n)$$

hold. To finish the proof note that $E \cap X_n \uparrow E$, and then use the "order continuity" of the measure (Theorem 15.4).

Problem 36.7. *If Σ is a σ-algebra of subsets of a set X and $\mu\colon \Sigma \to \mathbb{R}^*$ is a signed measure, then show that*

$$\Lambda_{\mu^+} \cap \Lambda_{\mu^-} = \Lambda_{|\mu|}.$$

Solution. Assume that $\mu\colon \Sigma \longrightarrow \mathbb{R}^*$ is an arbitrary signed measure. Let E be in $\Lambda_{\mu^+} \cap \Lambda_{\mu^-}$ and let $A \in \Sigma$ be an arbitrary set. Then,

$$\begin{aligned}
|\mu|(A) &= \mu^+(A) + \mu^-(A) \\
&= \big[\mu^+(A \cap E) + \mu^+(A \cap E^c)\big] + \big[\mu^-(A \cap E) + \mu^-(A \cap E^c)\big] \\
&= \big[\mu^+(A \cap E) + \mu^-(A \cap E)\big] + \big[\mu^+(A \cap E^c) + \mu^-(A \cap E^c)\big] \\
&= |\mu|(A \cap E) + |\mu|(A \cap E^c),
\end{aligned}$$

and so $E \in \Lambda_{|\mu|}$, i.e., $\Lambda_{\mu^+} \cap \Lambda_{\mu^-} \subseteq \Lambda_{|\mu|}$.

For the reverse inclusion, let $E \in \Lambda_{|\mu|}$. If $A \in \Sigma$ is arbitrary, then note that

$$\begin{aligned}
\mu^+(A) + \mu^-(A) &= |\mu|(A) = |\mu|(A \cap E) + |\mu|(A \cap E^c) \\
&= \big[\mu^+(A \cap E) + \mu^+(A \cap E^c)\big] + \big[\mu^-(A \cap E) + \mu^-(A \cap E^c)\big].
\end{aligned}$$

Since $\mu^+(A) = \mu^+\big((A \cap E) \cup (A \cap E^c)\big) \leq \mu^+(A \cap E) + \mu^+(A \cap E^c)$ and

$\mu^-(A) \le \mu^-(A \cap E) + \mu^-(A \cap E^c)$ both hold, it follows from the preceding equality that

$$\mu^+(A) = \mu^+(A \cap E) + \mu^+(A \cap E^c) \text{ and } \mu^-(A) = \mu^-(A \cap E) + \mu^-(A \cap E^c),$$

which shows that $E \in \Lambda_{\mu^+} \cap \Lambda_{\mu^-}$. Thus, $\Lambda_{|\mu|} \subseteq \Lambda_{\mu^+} \cap \Lambda_{\mu^-}$, and consequently, $\Lambda_{|\mu|} = \Lambda_{\mu^+} \cap \Lambda_{\mu^-}$ holds, as desired.

Problem 36.8. *Let μ and ν be two measures on a σ-algebra Σ with at least one of them finite. Assume also that $\mathcal{S}$ is a semiring such that $\mathcal{S} \subseteq \Sigma$, $X \in \mathcal{S}$, and that the σ-algebra generated by $\mathcal{S}$ equals Σ. Then show that $\mu = \nu$ on Σ if and only if $\mu = \nu$ on $\mathcal{S}$.*

Solution. Assume that μ is finite and that $\mu = \nu$ on $\mathcal{S}$. If we consider the measure space $(X, \mathcal{S}, \mu)$, then it is easy to see that $\mathcal{S} \subseteq \Sigma \subseteq \Lambda_\mu$ holds. Now, apply Theorem 15.10 to get that $\mu = \mu^* = \nu$ holds on Σ.

Problem 36.9. *Let $(X, \mathcal{S}, \mu)$ be a measure space, and let $f \in L_1(\mu)$. Then show that*

$$\nu(A) = \int_A f\,d\mu$$

for each $A \in \Lambda_\mu$ defines a finite signed measure on Λ_μ. Also, show that

$$\nu^+(A) = \int_A f^+\,d\mu, \quad \nu^-(A) = \int_A f^-\,d\mu \quad \text{and} \quad |\nu|(A) = \int_A |f|\,d\mu$$

holds for each $A \in \Lambda_\mu$.

Solution. If $\{A_n\}$ is a pairwise disjoint sequence of Λ_μ satisfying $A = \bigcup_{n=1}^\infty A_n$, then $\lim \sum_{i=1}^n f\chi_{A_i} = f\chi_A$ and $\left|\sum_{i=1}^n f\chi_{A_i}\right| \le |f|$ holds for each n. Thus, from the Lebesgue Dominated Convergence Theorem, it follows that

$$\nu(A) = \int f\chi_A\,d\mu = \lim_{n\to\infty} \int \Big(\sum_{i=1}^n f\chi_{A_i}\Big)\,d\mu = \lim_{n\to\infty} \sum_{i=1}^n \nu(A_i) = \sum_{i=1}^\infty \nu(A_i).$$

Therefore, ν is a finite signed measure.

Now, note that if

$$A = \{x \in X:\ f(x) \ge 0\} \quad \text{and} \quad B = \{x \in X:\ f(x) < 0\},$$

then it is easy to see that (A, B) is a Hahn decomposition of X with respect to ν. Since $f\chi_{E\cap A} = f^+\chi_E$ holds, we see that

$$\nu^+(E) = \nu(E \cap A) = \int_{E\cap A} f\,d\mu = \int_E f^+\,d\mu$$

for each $E \in \Lambda_\mu$. The proof for ν^- is similar. The absolute value formula follows from the identity $|\nu| = \nu^+ + \nu^-$.

Problem 36.10. *Let ν be a signed measure on Σ. A function $f: X \to \mathbf{R}$ is said to be ν-integrable if f is simultaneously ν^+- and ν^--integrable (in this case, we write $\int f\,d\nu = \int f\,d\nu^+ - \int f\,d\nu^-$). Show that a function f is ν-integrable if and only if $f \in L_1(|\nu|)$.*

Solution. Assume that f is simultaneously ν^+- and ν^--integrable. We can assume that $f(x) \geq 0$ holds for all $x \in X$. Since each set of the form $\{x \in X: a \leq f(x) < b\}$ belongs to $\Lambda_{\mu^+} \cap \Lambda_{\mu^-} = \Lambda_{|\mu|}$ (for this identity see Problem 36.7), we see that there exists a sequence $\{\phi_n\}$ of simultaneously ν^+- and ν^--step functions such that $\phi_n(x) \uparrow f(x)$ holds for all $x \in X$; see the proof of Theorem 17.7. Clearly, each ϕ_n is a $|\mu|$-step function and from

$$\int \phi_n\,d|\mu| = \int \phi_n\,d\mu^+ + \int \phi_n\,d\mu^- \uparrow \int f\,d\mu^+ + \int f d\mu^- < \infty,$$

we see that f is $|\mu|$-integrable and that $\int f\,d|\mu| = \int f\,d\mu^+ + \int f\,d\mu^-$ holds.

For the converse, assume that f belongs to $L_1(|\nu|)$. We can assume that $f(x) \geq 0$ holds for each x. Note first that if $f = \chi_A$ for a $|\nu|$-measurable set A with $|\nu|^*(A) < \infty$, then there exists (by Theorem 15.11) some $B \in \Sigma$ with $A \subseteq B$ and $|\nu|^*(A) = |\nu|^*(B)$. It follows that $|\nu|^*(B \setminus A) = 0$, and in view of $0 \leq \nu^+ \leq |\nu|$, we have $(\nu^+)^*(B \setminus A) = 0$. Thus, $B \setminus A$ is a ν^+-measurable set, and consequently, $A = B \setminus (B \setminus A)$ is also ν^+-measurable. This shows that χ_A is ν^+-integrable.

Now, choose a sequence $\{\phi_n\}$ of $|\nu|$-step functions with $0 \leq \phi_n(x) \uparrow f(x)$ for each x. By the previous discussion, $\{\phi_n\}$ is a sequence of ν^+-step functions. Moreover,

$$\int \phi_n\,d\nu^+ \leq \int \phi_n\,d|\nu| \leq \int f\,d|\nu| < \infty$$

holds for all n. Thus, f is ν^+-integrable.

The ν^--integrability of f can be established in a similar manner.

Problem 36.11. *Show that the Jordan decomposition is unique in the following sense. If ν is a signed measure, and μ_1 and μ_2 are two measures such that $\nu = \mu_1 - \mu_2$ and $\mu_1 \wedge \mu_2 = 0$, then $\mu_1 = \nu^+$ and $\mu_2 = \nu^-$.*

Solution. First, we shall establish that $\nu^+ = \mu_1$ holds. Start by observing that $\nu \le \mu_1$ implies $\nu^+ \le \mu_1$.

Now, let $E \in \Sigma$. If $\nu^+(E) = \infty$, then $\nu^+(E) = \mu_1(E) = \infty$ holds trivially. Thus, we can suppose $\nu^+(E) < \infty$. Since $\nu(E) = \mu_1(E) - \mu_2(E) \le \nu^+(E) < \infty$, it follows that $\mu_1(E) < \infty$. Let $\varepsilon > 0$. Then, in view of

$$0 = \mu_1 \wedge \mu_2(E) = \inf\{\mu_1(E \setminus B) + \mu_2(B)\colon\ B \in \Sigma \text{ and } B \subseteq E\},$$

there exists some $B \in \Sigma$ with $B \subseteq E$ and $\mu_1(E \setminus B) + \mu_2(B) < \varepsilon$. Thus,

$$\begin{aligned}\nu^+(E) &= \sup\{\nu(F)\colon\ F \in \Sigma \text{ and } F \subseteq E\} \ge \nu(B) = \mu_1(B) - \mu_2(B)\\ &\ge \mu_1(B) - \varepsilon = \mu_1(E) - \mu_1(E \setminus B) - \varepsilon \ge \mu_1(E) - 2\varepsilon\end{aligned}$$

holds for all $\varepsilon > 0$. That is, $\nu^+(E) \ge \mu_1(E)$ for each $E \in \Sigma$, and therefore $\nu^+ = \mu_1$ holds. For the other identity note that

$$\nu^- = (-\nu)^+ = (\mu_2 - \mu_1)^+ = \mu_2.$$

Problem 36.12. *In a vector lattice $x_n \downarrow x$ means that $x_{n+1} \le x_n$ for each n and that x is the greatest lower bound of the sequence $\{x_n\}$. A normed vector lattice is said to have σ-***order continuous norm** *if $x_n \downarrow 0$ implies $\lim \|x_n\| = 0$.*

a. *Show that every $L_p(\mu)$ with $1 \le p < \infty$ has σ-order continuous norm.*
b. *Show that $L_\infty([0, 1])$ does not have σ-order continuous norm.*
c. *Let Σ be a σ-algebra of sets, and let $\{\mu_n\}$ be a sequence of $M(\Sigma)$ such that $\mu_n \downarrow \mu$. Show that $\lim \mu_n(A) = \mu(A)$ holds for all $A \in \Sigma$.*
d. *Show that the Banach lattice $M(\Sigma)$ has σ-order continuous norm.*

Solution. (a) Note first that $f_n \downarrow f$ in $L_p(\mu)$ is equivalent to $f_n \downarrow f$ a.e. (why?). If for some $1 \le p < \infty$ a sequence $\{f_n\}$ of $L_p(\mu)$ satisfies $f_n \downarrow 0$ a.e., then

$$\|f_n\|_p = \Big(\int |f_n|^p \, d\mu\Big)^{\frac{1}{p}} \downarrow 0$$

holds by virtue of the Lebesgue Dominated Convergence Theorem.

(b) If $f_n = \chi_{(0,\frac{1}{n})}$, then $f_n \downarrow 0$ holds in $L_\infty([0, 1])$. However, note that $\|f_n\|_\infty = 1$ holds for each n.

(c) Let $\mu_n \downarrow \mu$ in $M(\Sigma)$. Then $0 \le \mu_1 - \mu_n \uparrow \mu_1 - \mu$ in $M(\Sigma)$. By Theorem 36.2, it follows that $\mu_1(A) - \mu_n(A) \uparrow \mu_1(A) - \mu(A)$ holds for each $A \in \Sigma$. Thus, $\mu_n(A) \downarrow \mu(A)$ holds for each $A \in \Sigma$.
(d) If $\mu_n \downarrow 0$ in $M(\Sigma)$, then from part (c) it follows that $\|\mu_n\| = \mu_n(X) \downarrow 0$.

Problem 36.13. *Prove the following additivity property of the Banach lattice $M(\Sigma)$: If $\mu, \nu \in M(\Sigma)$ are disjoint (i.e., $|\mu| \wedge |\nu| = 0$), then $\|\mu+\nu\| = \|\mu\|+\|\nu\|$ holds.*

Solution. If $|\mu| \wedge |\nu| = 0$ holds in $M(\Sigma)$, then $|\mu + \nu| = |\mu| + |\nu|$ holds (see Problems 9.2 and 9.3). Thus, $\|\mu + \nu\| = |\mu + \nu|(X) = |\mu|(X) + |\nu|(X) = \|\mu\| + \|\nu\|$.

Problem 36.14. *Let Σ be a σ-algebra of subsets of a set X and let $\{\mu_n\}$ be a disjoint sequence of $M(\Sigma)$. If the sequence of signed measures $\{\mu_n\}$ is order bounded, then show that* $\lim \|\mu_n\| = 0$.

Solution. Let $\{\mu_n\}$ be a disjoint sequence of the Banach lattice $M(\Sigma)$ such that for some $0 \le \mu \in M(\Sigma)$ we have $|\mu_n| \le \mu$ for each n. From $|\mu_n| \wedge |\mu_m| = 0$ for $n \ne m$, we see that

$$\sum_{n=1}^{k} |\mu_n| = \bigvee_{n=1}^{k} |\mu_n| \le \mu$$

holds for each k. In particular, we have

$$\sum_{n=1}^{k} \|\mu_n\| = \sum_{n=1}^{k} |\mu_n|(X) \le \Big[\bigvee_{n=1}^{k} |\mu_n|\Big](X) \le \mu(X) < \infty$$

holds for each n, and so $\sum_{n=1}^{\infty} \|\mu_n\| < \infty$. The latter easily implies $\lim \|\mu_n\| = 0$.

37. COMPARING MEASURES AND THE RADON–NIKODYM THEOREM

Problem 37.1. *Verify the following properties of signed measures:*

a. $\mu \ll \mu$.
b. *$\nu \ll \mu$ and $\mu \ll \omega$ imply $\nu \ll \omega$.*
c. *If $0 \le \nu \le \mu$, then $\nu \ll \mu$.*
d. *If $\mu \ll 0$, then $\mu = 0$.*

Solution. (a) From Theorem 36.9, we have $|\mu(A)| \leq |\mu|(A)$, and so if $|\mu|(A) = 0$ holds, then $\mu(A) = 0$ likewise holds. That is, $\mu \ll \mu$.
(b) Assume $\nu \ll \mu$ and $\mu \ll \omega$ and $|\omega|(A) = 0$. Theorem 37.2 applied twice shows that $|\mu|(A) = 0$ and $|\nu|(A) = 0$. Hence, $\nu \ll \omega$ holds.
(c) Let $0 \leq \nu \leq \mu$. If $|\mu|(A) = \mu(A) = 0$, then clearly $\nu(A) = 0$, and so $\nu \ll \mu$ holds.
(d) Let $\mu \ll 0$. Since the zero measure assumes the zero value at every $A \in \Sigma$, it follows that $\mu(A) = 0$ holds for every $A \in \Sigma$. This means that $\mu = 0$.

Problem 37.2. *Verify the following statements about signed measures on a σ-algebra Σ of sets:*

1. *If $\mu \ll \omega$ and $\nu \ll \omega$, then $|\mu| + |\nu| \ll \omega$.*
2. *If $\mu \perp \omega$ and $\nu \perp \omega$, then $|\mu| + |\nu| \perp \omega$.*
3. *If $\mu \ll \omega$ and $|\nu| \leq |\mu|$, then $\nu \ll \omega$.*
4. *If $\mu \perp \omega$ and $|\nu| \leq |\mu|$, then $\nu \perp \omega$.*
5. *If $\nu \ll \mu$ and $\nu \perp \mu$, then $\nu = 0$.*

Solution. (1) This follows immediately from Theorem 37.2.
(2) Since $\mu \perp \omega$, there exists (by Theorem 37.5) some $A_1 \in \Sigma$ with $|\omega|(A_1) = |\mu|(A_1^c) = 0$. Similarly, there exists some $A_2 \in \Sigma$ with $|\omega|(A_2) = |\nu|(A_2^c) = 0$. Put $A = A_1 \cup A_2$ and $B = (A_1 \cup A_2)^c = A_1^c \cap A_2^c$. Then $A,\ B \in \Sigma$, $A \cup B = X$, $A \cap B = \emptyset$, $|\omega|(A) = 0$, and $(|\mu| + |\nu|)(B) = 0$. By Theorem 37.5 we infer that $\omega \perp |\mu| + |\nu|$ holds.
(3) This follows easily from Theorem 37.2.
(4) This follows immediately from Theorem 37.5.
(5) Since $\nu \perp \mu$, there exists some $A \in \Sigma$ such that $|\nu|(A) = |\mu|(A^c) = 0$. By $\nu \ll \mu$ and Theorem 37.2, $|\nu|(A^c) = 0$, and so $|\nu|(X) = |\nu|(A) + |\nu|(A^c) = 0$. That is, $|\nu| = 0$, so that $\nu = 0$.

Problem 37.3. *Let μ and ν be two measures on a σ-algebra Σ. If ν is a finite measure, then show that the following statements are equivalent.*

a. *$\nu \ll \mu$ holds.*
b. *For each sequence $\{A_n\}$ of Σ with $\lim \mu(A_n) = 0$, we have $\lim \nu(A_n) = 0$.*
c. *For each $\epsilon > 0$ there exists some $\delta > 0$ (depending on ϵ) such that whenever $A \in \Sigma$ satisfies $\mu(A) < \delta$, then $\nu(A) < \epsilon$ holds.*

Solution. (a) $\Longrightarrow$ (b) If (b) is not true, then there exists some $\varepsilon > 0$ and some sequence $\{A_n\}$ of Σ such that $\mu(A_n) < 2^{-n}$ and $\nu(A_n) > \varepsilon$ for each n. Set

$A = \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} A_i \in \Sigma$. From $A \subseteq \bigcup_{i=n}^{\infty} A_i$, we see that

$$\mu(A) \leq \sum_{i=n}^{\infty} \mu(A_i) \leq \sum_{i=n}^{\infty} \tfrac{1}{2^n} = 2^{1-n}$$

holds for each n, and so $\mu(A) = 0$. However, from Theorem 15.4(2), we see that $\nu(A) \geq \varepsilon$, contrary to $\nu \ll \mu$. Hence, (a) implies (b).
(b)$\Longrightarrow$(c) If (c) is not true, then there exist some $\varepsilon > 0$ and a sequence $\{A_n\}$ of Σ such that $\mu(A_n) < \frac{1}{n}$ and $\nu(A_n) \geq \varepsilon$ hold for all n. Clearly, this contradicts (b).
(c)$\Longrightarrow$(a) Let $A \in \Sigma$ satisfy $\mu(A) = 0$. Given $\varepsilon > 0$, choose some $\delta > 0$ so that (c) is satisfied. In view of $\mu(A) < \delta$, it follows that $\nu(A) < \varepsilon$. Since $\varepsilon > 0$ is arbitrary, $\nu(A) = 0$, and so $\nu \ll \mu$ holds.

Problem 37.4. *Let μ be a finite measure, and let $\{\nu_n\}$ be a sequence of finite measures (all on Σ) such that $\nu_n \ll \mu$ holds for each n. Furthermore, assume that* lim $\nu_n(A)$ *exists in* $\mathbb{R}$ *for each $A \in \Sigma$. Then, show that:*

a. *For each $\epsilon > 0$ there exists some $\delta > 0$ such that whenever $A \in \Sigma$ satisfies $\mu(A) < \delta$, then $\nu_n(A) < \epsilon$ holds for each n.*
b. *The set function $\nu\colon \Sigma \to [0, \infty]$, defined by $\nu(A) = \lim \nu_n(A)$ for each $A \in \Sigma$, is a measure such that $\nu \ll \mu$.*

Solution. (a) From Problem 31.3, we know that Σ under the distance $d(A, B) = \mu(A \Delta B)$ is a complete metric space. From $\nu_k \ll \mu$ and the inequality

$$\big|\nu_k(A) - \nu_k(B)\big| \leq \nu_k(A \Delta B),$$

it easily follows that the function $\nu_k\colon \Sigma \longrightarrow \mathbb{R}$ is well defined (i.e., $\nu_k(A) = \nu_k(B)$ holds whenever $\mu(A \Delta B) = 0$) and is continuous.

Now, let $\varepsilon > 0$. Define

$$C_k = \Big\{A \in \Sigma\colon |\nu_n(A) - \nu_m(A)| \leq \varepsilon \text{ for all } n,\ m \geq k\Big\}.$$

Note that each C_k is closed and that $\Sigma = \bigcup_{k=1}^{\infty} C_k$ holds. By Baire's Category Theorem 6.18), we have $C_k^{\circ} \neq \emptyset$ for some k. Thus, there exist $A_0 \in C_k$ and $\delta_1 > 0$ such that $A \in \Sigma$ and $\mu(A \Delta A_0) < \delta_1$ imply $A \in C_k$.

From $\nu_i \ll \mu$ $(1 \leq i \leq k)$ and the preceding problem, there exists some $0 < \delta < \delta_1$ such that $A \in \Sigma$ and $\mu(A) < \delta$ imply $\nu_i(A) < \varepsilon$ for all $1 \leq i \leq k$.

Now, if $A \in \Sigma$ satisfies $\mu(A) < \delta$, then $A \cup (A_0 \setminus A) = A \cup A_0$ satisfies $\mu\big((A \cup A_0)\Delta A_0\big) \le \mu(A) < \delta_1$, and so

$$\begin{aligned}\big|\nu_n(A) - \nu_k(A)\big| &= \big|(\nu_n - \nu_k)(A \cup A_0) - (\nu_n - \nu_k)(A_0 \setminus A)\big| \\ &\le \big|(\nu_n - \nu_k)(A \cup A_0)\big| + \big|(\nu_n - \nu_k)(A_0 \setminus A)\big| \le 2\varepsilon\end{aligned}$$

holds for all $n > k$. Thus, $A \in \Sigma$ and $\mu(A) < \delta$ imply

$$\big|\nu_n(A)\big| \le 2\varepsilon + \nu_k(A) < 3\varepsilon$$

for all $n > k$ (and all $1 \le n \le k$).

(b) Let $A = \bigcup_{n=1}^{\infty} A_n$ with the sequence $\{A_n\}$ of Σ pairwise disjoint, and let $\varepsilon > 0$. Choose some $\delta > 0$ so that statement (a) is satisfied. Next, choose some m so that $\mu\big(A \setminus \bigcup_{i=1}^{n} A_i\big) < \delta$ holds for all $n \ge m$. Then,

$$\Big|\nu_k(A) - \sum_{i=1}^{n} \nu_k(A_i)\Big| = \nu_k\Big(A \setminus \bigcup_{i=1}^{n} A_i\Big) < \varepsilon$$

holds for all k and all $n \ge m$. Thus, $\big|\nu(A) - \sum_{i=1}^{n} \nu(A_i)\big| \le \varepsilon$ holds for all $n \ge m$, and so $\big|\nu(A) - \sum_{i=1}^{\infty} \nu(A_i)\big| \le \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we see that $\nu(A) = \sum_{n=1}^{\infty} \nu(A_n)$. Thus, ν is a measure, and from part (a) and the preceding problem it follows immediately that $\nu \ll \mu$ holds.

Problem 37.5. *Let $\{\nu_n\}$ be a sequence of nonzero finite measures such that* $\lim \nu_n(A)$ *exists in* $\mathbf{R}$ *for each $A \in \Sigma$. Show that* $\nu(A) = \lim \nu_n(A)$ *for $A \in \Sigma$ is a finite measure.*

Solution. Consider the set function $\mu\colon \Sigma \longrightarrow [0, \infty)$ defined by

$$\mu(A) = \sum_{n=1}^{\infty} \tfrac{\nu_n(A)}{\nu_n(X)}\, 2^{-n},$$

and note that μ is in fact a measure. In addition, note that $\nu_n \ll \mu$ holds for each n. Now, invoke part (b) of the preceding problem to conclude that the set function ν is also a measure.

Problem 37.6. *Verify the uniqueness of the Radon–Nikodym derivative by proving the following statement: If $(X, \mathcal{S}, \mu)$ is a measure space and $f \in L_1(\mu)$ satisfies $\int_A f\,d\mu = 0$ for all $A \in \mathcal{S}$, then $f = 0$ a.e.*

Solution. From the given condition, it is easy to see that $\int_A f\,d\mu = 0$ must hold for each σ-set A. Now, consider the measurable sets

$$A = \{x \in X\colon\ f(x) > 0\} \quad \text{and} \quad B = \{x \in X\colon\ f(x) < 0\}.$$

By Problem 22.7 we know that A and B are both σ-finite sets. Now, in view of $\int_A f\,d\mu = \int_B f\,d\mu = 0$, it follows from Problem 22.13 that $\mu^*(A) = \mu^*(B) = 0$. Therefore, $f = 0$ a.e. holds.

Problem 37.7. *This problem shows that the hypothesis of σ-finiteness of μ in the Radon–Nikodym Theorem cannot be omitted. Consider $X = [0, 1]$, Σ the σ-algebra of all Lebesgue measurable subsets of $[0, 1]$, ν the Lebesgue measure on Σ and μ the measure defined by $\mu(\emptyset) = 0$ and $\mu(A) = \infty$ if $A \neq \emptyset$. (Incidentally, μ is the largest measure on Σ.) Show that:*

a. *ν is a finite measure, μ is not σ-finite, and $\nu \ll \mu$.*
b. *There is no function $f \in L_1(\mu)$ such that $\nu(A) = \int_A f\,d\mu$ holds for all $A \in \Sigma$.*

Solution. (a) Note that $\mu(A) = 0$ means $A = \emptyset$, and so $\nu \ll \mu$ holds.
(b) Observe that $L_1(\mu) = \{0\}$.

Problem 37.8. *Let μ be a finite signed measure on Σ. Show that there exists a unique function $f \in L_1(|\mu|)$ such that*

$$\mu(A) = \int_A f\,d|\mu|$$

holds for all $A \in \Sigma$.

Solution. The conclusion follows from the Radon–Nikodym Theorem by observing that $\mu \ll |\mu|$ holds.

Problem 37.9. *Assume that ν is a finite measure and μ is a σ-finite measure such that $\nu \ll \mu$. Let $g = d\nu/d\mu \in L_1(\mu)$ be the Radon–Nikodym derivative of ν with respect to μ. Then show that:*

a. *If $Y = \{x \in X\colon\ g(x) > 0\}$, then $Y \cap A$ is a μ-measurable set for each ν-measurable set A.*
b. *If $f \in L_1(\nu)$, then $fg \in L_1(\mu)$ and $\int f\,d\nu = \int fg\,d\mu$ holds.*

Solution. (a) Note first that by Theorem 37.3, $\Sigma \subseteq \Lambda_\mu \subseteq \Lambda_\nu$ holds, and that $Y \in \Lambda_\mu$.

First consider the case when $A \in \Lambda_\nu$ satisfies $A \subseteq Y$ and $\nu^*(A) = 0$. By Theorem 15.11 there exists some $B \in \Sigma$ with $A \subseteq B$ and $\nu^*(B) = 0$. Now, if $\mu^*(B \cap Y) > 0$, then we have the contradiction $0 = \nu^*(B \cap Y) = \int_{B\cap Y} g\,d\mu > 0$ (see Problem 22.13). Consequently, $\mu^*(B \cap Y) = \mu^*(A) = 0$ holds, and so $A \in \Lambda_\mu$.

Now, let $A \in \Lambda_\nu$. Choose some $B \in \Sigma$ with $A \subseteq B$ and $\nu^*(A) = \nu^*(B)$. Thus, $\nu^*(B \setminus A) = 0$, and so $(B \setminus A) \cap Y \in \Lambda_\mu$. Now, note that

$$A \cap Y = B \cap Y \setminus (B \setminus A) \cap Y \in \Lambda_\mu.$$

(b) It follows immediately from Problem 22.15.

Problem 37.10. *Establish the* **chain rule** *for Radon–Nikodym derivatives: If ω is a σ-finite measure and ν and μ are two finite measures (all on Σ) such that $\nu \ll \mu$ and $\mu \ll \omega$, then $\nu \ll \omega$ and*

$$\frac{d\nu}{d\omega} = \frac{d\nu}{d\mu} \cdot \frac{d\mu}{d\omega} \quad (\omega\text{-a.e.})$$

holds.

Solution. Clearly, $\nu \ll \mu$ and $\Sigma \subseteq \Lambda_\omega \subseteq \Lambda_\mu \subseteq \Lambda_\nu$. Put $f = \frac{d\nu}{d\mu} \in L_1(\mu)$ and $g = \frac{d\mu}{d\omega} \in L_1(\omega)$. If $A \in \Sigma$, then by part (b) of the preceding problem, we infer that

$$\nu(A) = \int_A f\,d\mu = \int f\chi_A\,d\mu = \int f\chi_A g\,d\omega = \int_A fg\,d\omega.$$

This combined with the Radon–Nikodym Theorem shows that

$$\frac{d\nu}{d\mu} = fg, \ \omega\text{–}a.e.$$

Problem 37.11. *All measures considered here will be assumed defined on a fixed σ-algebra Σ.*

a. *Call two measures μ and ν equivalent (in symbols, $\mu \equiv \nu$) if $\mu \ll \nu$ and $\nu \ll \mu$ both hold. Show that $\equiv$ is an equivalence relation among the measures on Σ.*

b. *If μ and ν are two equivalent σ-finite measures, then show that $\Lambda_\mu = \Lambda_\nu$.*

c. *Show that if μ and ν are two equivalent finite measures, then*

$$\frac{d\mu}{d\nu} \cdot \frac{d\nu}{d\mu} = 1 \ \text{a.e. holds.}$$

d. *If μ and ν are two equivalent finite measures, then show that*

$$f \mapsto f \cdot \frac{d\mu}{d\nu},$$

from $L_1(\mu)$ to $L_1(\nu)$, is an onto lattice isometry. Thus, under this identification $L_1(\mu) = L_1(\nu)$ holds.

e. *Generalize (d) to equivalent σ-finite measures. That is, if μ and ν are two equivalent σ-finite measures, then show that the Banach lattices $L_1(\mu)$ and $L_1(\nu)$ are lattice isometric.*

f. *Show that if μ and ν are two equivalent σ-finite measures, then the Banach lattices $L_p(\mu)$ and $L_p(\nu)$ are lattice isometric for each $1 \le p \le \infty$.*

Solution. (a) Straightforward.
(b) It follows immediately from Theorem 37.3.
(c) Use the relation $\nu \ll \mu \ll \nu$ and the preceding problem.
(d) Let $f \longmapsto f \cdot \frac{d\mu}{d\nu} = T(f)$. Since $\frac{d\mu}{d\nu} \in L_1(\nu)$, it follows from Problem 37.9(b) that $T(f) = f \cdot \frac{d\mu}{d\nu} \in L_1(\nu)$ and $\int f\,d\mu = \int f \cdot \frac{d\mu}{d\nu}\,d\nu$ hold for each $f \in L_1(\mu)$. Thus, T defines a mapping from $L_1(\mu)$ to $L_1(\nu)$ which is clearly linear. Since $\frac{d\mu}{d\nu} \ge 0$ holds, it follows that

$$T(|f|) = |f| \cdot \tfrac{d\mu}{d\nu} = \left|f \cdot \tfrac{d\mu}{d\nu}\right| = |T(f)|$$

and

$$\|T(f)\|_1 = \int \left|f \cdot \tfrac{d\mu}{d\nu}\right| d\nu = \int |f| \cdot \tfrac{d\mu}{d\nu}\,d\nu = \int |f|\,d\mu = \|f\|_1$$

hold for each $f \in L_1(\mu)$. Thus, $T\colon L_1(\mu) \longrightarrow L_1(\nu)$ is a lattice isometry. To see that T is also onto, note that if $g \in L_1(\nu)$, then $g \cdot \frac{d\nu}{d\mu} \in L_1(\mu)$ and by part (c), we see that

$$T\left(g \cdot \tfrac{d\nu}{d\mu}\right) = g \cdot \tfrac{d\nu}{d\mu} \cdot \tfrac{d\mu}{d\nu} = g.$$

(e) Let $\{E_n\}$ be a pairwise disjoint sequence of Σ such that $\bigcup_{n=1}^{\infty} E_n = X$, $\mu(E_n) < \infty$, and $\nu(E_n) < \infty$ for each n. Let

$$T_n\colon L_1(E_n, \mu) \longrightarrow L_1(E_n, \nu)$$

be the onto lattice isometry determined by part (d) previously. Now, it is a routine matter to verify that $T\colon L_1(\mu) \longrightarrow L_1(\nu)$ defined by

$$T(f) = \sum_{n=1}^{\infty} T_n(f\chi_{E_n})$$

for each $f \in L_1(\mu)$ is an onto lattice isometry.

(f) Suppose first that μ and ν are finite. Then,

$$f \longmapsto f \cdot \left(\tfrac{d\mu}{d\nu}\right)^{\frac{1}{p}}$$

is a lattice isometry from $L_p(\mu)$ onto $L_p(\nu)$ for each $1 \le p < \infty$. Now, if μ and ν are σ-finite, then use the arguments of part (e) to establish that $L_p(\mu)$ and $L_p(\nu)$ are lattice isometric.

If $p = \infty$, then from part (b) it follows that $L_\infty(\mu) = L_\infty(\nu)$ holds, and so in this case the identity operator is a lattice isometry.

Problem 37.12. *Let μ be a σ-finite measure, and let $AC(\mu)$ be the collection of all finite signed measures that are absolutely continuous with respect to μ; that is,*

$$AC(\mu) = \{\nu \in M(\Sigma)\colon\ \nu \ll \mu\}.$$

a. *Show that $AC(\mu)$ is a norm closed ideal of $M(\Sigma)$ (and hence $AC(\mu)$, with the norm $\|\nu\| = |\nu|(X)$, is a Banach lattice in its own right).*
b. *For each $f \in L_1(\mu)$, let μ_f be the finite signed measure defined by $\mu_f(A) = \int_A f\,d\mu$ for each $A \in \Sigma$. Then show that $f \mapsto \mu_f$ is a lattice isometry from $L_1(\mu)$ onto $AC(\mu)$.*

Solution. (a) Clearly, $AC(\mu)$ is an ideal of $M(\Sigma)$. If $\{\nu_n\}$ is a sequence of $AC(\mu)$ satisfying $\nu_n \longrightarrow \nu$ in $M(\Sigma)$, then $\nu_n(A) \longrightarrow \nu(A)$ holds for each $A \in \Sigma$. Problem 37.4 shows that $\nu \in AC(\mu)$. Thus, $AC(\mu)$ is a closed vector sublattice of $M(\Sigma)$, and hence, a Banach lattice in its own right.
(b) Clearly, $f \longmapsto \mu_f$ is a linear operator. By Problem 37.6 this operator is one-to-one. From Problem 36.9, it follows that $f \longmapsto \mu_f$ is a lattice isometry, and the Radon–Nikodym Theorem implies that it is also onto.

Problem 37.13. *Let Σ be a σ-algebra of subsets of a set X and μ a measure on Σ. Assume also that Σ^* is a σ-algebra of subsets of a set Y and that $T\colon X \to Y$ has the property that $T^{-1}(A) \in \Sigma$ for each $A \in \Sigma^*$.*

a. *Show that $\nu(A) = \mu(T^{-1}(A))$ for each $A \in \Sigma^*$ is a measure on Σ^*.*
b. *If $f \in L_1(\nu)$, then show that $f \circ T \in L_1(\mu)$ and*

$$\int_Y f\,d\nu = \int_X f \circ T\,d\mu.$$

c. *If μ is finite and ω is a σ-finite measure on Σ^* such that $\nu \ll \omega$, then show that there exists a function $g \in L_1(\omega)$ such that*

$$\int_X f \circ T\,d\mu = \int_Y fg\,d\omega$$

holds for each $f \in L_1(\nu)$.

Solution. (a) Straightforward.
(b) Note first that if A is a ν-null set, then $T^{-1}(A)$ is a μ-null set. Indeed, if $\nu^*(A) = 0$ holds, then there exists (by Theorem 15.11) some $B \in \Sigma^*$ with $A \subseteq B$ and $\nu(B) = 0$. Therefore,

$$0 \le \mu^*\big(T^{-1}(A)\big) \le \mu^*\big(T^{-1}(B)\big) = \mu\big(T^{-1}(B)\big) = \nu(B) = 0.$$

Now, let A be a ν-measurable set with $\nu^*(A) < \infty$. Choose some $B \in \Sigma^*$ with $A \subseteq B$ and $\nu^*(B) = \nu^*(A)$. Since $\nu^*(B \setminus A) = 0$, it follows from the preceding discussion that $\mu^*\big(T^{-1}(B \setminus A)\big) = 0$. Thus, $T^{-1}(A) = T^{-1}(B) \setminus T^{-1}(B \setminus A)$ is μ-measurable, and moreover,

$$\int_Y \chi_A \, d\nu = \nu^*(A) = \mu^*\big(T^{-1}(A)\big) = \int_X \chi_{T^{-1}(A)} \, d\mu = \int_X \chi_A \circ T \, d\mu.$$

It follows that for every ν-step function ϕ we have $\phi \circ T \in L_1(\mu)$, and $\int_Y \phi \, d\nu = \int_X \phi \circ T \, d\mu$. An easy continuity argument can complete the proof.
(c) It follows immediately from part (b) and Problem 37.9.

Problem 37.14. *Let (X, S, μ) be a σ-finite measure space, and let g be a measurable function. Show that if for some $1 \le p < \infty$ we have $fg \in L_1(\mu)$ for all $f \in L_p(\mu)$, then $g \in L_q(\mu)$, where $\frac{1}{p} + \frac{1}{q} = 1$.*
Also, show by a counterexample that for $1 < p < \infty$ the σ-finiteness of μ cannot be dropped.

Solution. We can assume that $g \ge 0$ holds (why?). Then the formula $F(f) = \int fg \, d\mu$ for $f \in L_p(\mu)$ defines a positive linear functional on $L_p(\mu)$. By Theorem 40.10, F is continuous. Now, by Theorems 37.9 and 37.10 there exists some $h \in L_q(\mu)$ such that $\int fg \, d\mu = \int fh \, d\mu$ for each $f \in L_p(\mu)$. This implies (how?) $\int_A (g - h) \, d\mu = 0$ for each measurable subset A. Now, a glance at Problem 22.13 guarantees that $g = h$ a.e. holds.

The σ-finiteness of μ cannot be dropped. Consider $X = (0, \infty)$ with the measure μ defined on the σ-algebra $\mathcal{P}(X)$ by $\mu(A) = \infty$ if $A \ne \emptyset$ and $\mu(\emptyset) = 0$. Then for $1 < p < \infty$ we have $L_1(\mu) = L_p(\mu) = L_q(\mu) = \{0\}$, and $L_\infty(\mu) = B(X)$, the bounded real-valued functions on X. On the other hand, if $g(x) = x$, then $fg = 0 \in L_1(\mu)$ holds for all $f \in L_p(\mu)$ $(1 \le p < \infty)$, while $g \notin L_q(\mu)$.

Problem 37.15. *Let (X, S, μ) be a σ-finite measure space, g a measurable function, and $1 \le p < \infty$. Assume that there exists some real number $M > 0$ such that $\phi g \in L_1(\mu)$ and $\int \phi g \, d\mu \le M \|\phi\|_p$ holds for every step function ϕ. Then, show that:*

a. $g \in L_q(\mu)$, *where* $\frac{1}{p} + \frac{1}{q} = 1$, *and*
b. $\int fg\,d\mu \le M\|f\|_q$ *holds for all* $f \in L_p(\mu)$.

Solution. Let L denote the vector space of all step functions. The given conditions show that the function $F: L \longrightarrow \mathbf{R}$, defined by $F(\phi) = \int \phi g\,d\mu$, is a continuous linear functional. Since L is dense in $L_p(\mu)$ (Theorem 31.10), it follows that F has a continuous extension (which we shall denote by F again) to all of $L_p(\mu)$. By Theorems 37.9 and 37.10 there exists some $h \in L_q(\mu)$ such that $F(f) = \int fh\,d\mu$ holds for all $f \in L_p(\mu)$. Clearly,

$$\left|F(f)\right| = \left|\int fh\,d\mu\right| \le M\|f\|_p$$

holds for all $f \in L_p(\mu)$.

To complete the proof, it suffices to show that $g = h$ a.e. holds. To see this, let $E \in \Lambda_\mu$ satisfy $\mu^*(E) < \infty$. Then, consider the step function $\phi = \chi_E \operatorname{Sgn}(g-h) \in L$, and note that $\int \phi(g-h)\,d\mu = 0$ implies $\int_E |g-h|\,d\mu = 0$. That is, $g = h$ a.e. holds on E; see Problem 22.13. Since μ is σ-finite, we see that $g = h$ a.e. holds on X.

Problem 37.16. *Let μ be a Borel measure on $\mathbf{R}^k$ and suppose that there exists a constant $c > 0$ such that whenever a Borel set E satisfies $\lambda(E) = c$, then $\mu(E) = c$. Show that μ coincides with λ, i.e., show that $\mu = \lambda$.*

Solution. Assume that the Borel measure μ and the constant $c > 0$ satisfy the properties of the problem. Clearly, μ is a σ-finite Borel measure. By Theorem 37.7, we can write

$$\mu = \mu_1 + \mu_2, \quad \text{where } \mu_1 \ll \lambda \text{ and } \mu_2 \perp \lambda.$$

First, we shall establish that $\mu_2 = 0$. From $\mu_2 \perp \lambda$, there exist two disjoint Borel sets A and B with $A \cup B = \mathbf{R}^k$ and $\mu_2(A) = \lambda(B) = 0$. Since $\lambda(A) = \infty$, there exists (by Problem 18.19) a Borel subset C of A with $\lambda(C) = c$. From $\lambda(C \cup B) = \lambda(C) + \lambda(B) = \lambda(C) = c$ and our hypothesis, we see that $\mu(C \cup B) = c$. Now, note that

$$c \le c + \mu_2(B) \le c + \mu(B) = \mu(C) + \mu(B) = \mu(C \cup B) = c,$$

and so $\mu_2(B) = 0$. This shows that $\mu_2 = 0$, and consequently $\mu = \mu_1$ is absolutely continuous with respect to λ.

Next, fix a compact set K with $\lambda(K) \geq c$ and consider both μ and λ restricted to K. By the Radon–Nikodym Theorem, there exists a non-negative function $f \in L_1(K, \mathcal{B}, \lambda)$ such that

$$\mu(E) = \int_E f\, d\lambda$$

holds for each Borel subset E of K (see Problem 12.13). We claim that $f = \mathbf{1}$ a.e. To establish this, assume by way of contradiction that the Lebesgue measurable set $D = \{x \in K\colon\ f(x) < 1\}$ satisfies $\lambda(D) > 0$; we can assume (why?) that D is a Borel set. If $\lambda(D) \geq c$ holds, then pick a Borel subset D_1 of D with $\lambda(D_1) = c$; if $\lambda(D) < c$, then pick a Borel set D_1 with $D \subseteq D_1 \subseteq K$ and $\lambda(D_1) = c$; (see Problem 18.19). Now, note that in either case, we have $\mu(D_1) < c$, which contradicts our hypothesis. Hence, $\lambda(D) = 0$. Similarly, $\lambda(\{x \in K\colon\ f(x) > 1\} = 0$, and so $f = \mathbf{1}$ a.e. Therefore, $\mu(E) = \lambda(E)$ holds for each Borel subset E of K. Now, pick a sequence $\{K_n\}$ of compact subsets of $\mathbf{R}^k$ with $\lambda(K_n) \geq c$ and $K_n \uparrow \mathbf{R}^k$. If E is an arbitrary Borel subset of $\mathbf{R}^k$, then note that

$$\mu(E) = \lim_{n\to\infty} \mu(E \cap K_n) = \lim_{n\to\infty} \lambda(E \cap K_n) = \lambda(E).$$

Problem 37.17. *Let μ and ν be two σ-finite measures on a σ-algebra Σ of subsets of a set X such that $\nu \ll \mu$ and $\nu \neq 0$. Show that there exist a set $E \in \Sigma$ and an integer n such that*

a. $\nu(E) > 0$; *and*
b. $A \in \Sigma$ *and* $A \subseteq E$ *imply* $\frac{1}{n}\mu(A) \leq \nu(A) \leq n\mu(A)$.

Solution. Pick a sequence $\{X_n\}$ of Σ with $X = \bigcup_{n=1}^{\infty} X_n$, $\nu(X_n) < \infty$, and $\mu(X_n) < \infty$ for each n. Since $\nu \neq 0$, there exists some n such that $\nu(X_n) > 0$. From $\nu \ll \mu$, it follows that $\mu(X_n) > 0$ also holds. Thus, replacing X by X_n, we can assume from the outset that both ν and μ are finite measures.

Now, by the Radon–Nikodym Theorem, there exists a function $0 \leq f \in L_1(\mu)$ such that

$$\nu(A) = \int_A f\, d\mu$$

holds for each $A \in \Sigma$. From $\nu \neq 0$, we see that $f \neq 0$, and so the μ-measurable set $F = \{x \in X\colon\ f(x) > 0\}$ satisfies $\mu^*(F) > 0$. Next, put

$$E_n = \left\{x \in X\colon\ \tfrac{1}{n} \leq f(x) \leq n\right\}$$

and note that $E_n \uparrow E$ a.e. Thus, for some n, we have $\mu^*(E_n) > 0$. By Theorem 15.11, there exists some $E \in \Sigma$ with $E_n \subseteq E$ and $\mu(E) = \mu^*(E_n)$. We claim that the set E satisfies the desired properties.

To see this, note first that

$$\nu(E) = \int_E f\, d\mu = \int_{E_n} f\, d\mu \geq \tfrac{1}{n}\mu^*(E_n) > 0.$$

Now, if $A \in \Sigma$ satisfies $A \subseteq E$, then note that $\frac{1}{n}\chi_A \leq f \leq n\chi_A$ μ – a.e., and consequently

$$\tfrac{1}{n}\mu(A) = \tfrac{1}{n}\int_A \chi_A\, d\mu \leq \int_A f\, d\mu = \nu(A) \leq \int_A n\chi_A\, d\mu = n\mu(A).$$

Problem 37.18. *Let μ be a finite Borel measure on $[1, \infty)$ such that*

a. $\mu \ll \lambda$, *and*
b. $\mu(B) = a\mu(aB)$ *for each $a \geq 1$ and each Borel subset B of $[1, \infty)$, where $aB = \{ab\colon b \in B\}$.*

If the Radon–Nikodym derivative $d\mu/d\lambda$ is a continuous function, then show that there exists a constant $c \geq 0$ such that $[d\mu/d\lambda](x) = \frac{c}{x^2}$ for each $x \geq 1$.

Solution. For simplicity, let us write $\frac{d\mu}{d\lambda} = f$. Then, the given identity $\mu(B) = a\mu(aB)$ can be written in the form

$$\int_B f\, d\lambda = a\int_{aB} f\, d\lambda.$$

For $B = [1, x]$, we get

$$\int_1^x f(t)\, dt = a\int_a^{ax} f(t)\, dt$$

for each $a \geq 1$ and each $x \geq 1$. Differentiating with respect to x (and taking into account the Fundamental Theorem of Calculus), we see that

$$f(x) = a^2 f(ax)$$

holds for each $x \geq 1$ and each $a \geq 1$. Letting $x = 1$, we obtain

$$f(a) = \tfrac{f(1)}{a^2}$$

for all $a \geq 1$, and our conclusion follows.

Problem 37.19. *Let μ be a finite Borel measure on $(0, \infty)$ such that*

a. $\mu \ll \lambda$, *and*
b. $\mu(aB) = \mu(B)$ *for each $a > 0$ and each Borel subset B of $(0, \infty)$.*

If the Radon–Nikodym derivative is a continuous function, then show that there exists a constant $c \geq 0$ such that $[d\mu/d\lambda](x) = \frac{c}{x}$ for each $x > 0$.

Solution. Let $\frac{d\mu}{d\lambda} = f$. Then, (by The Radon–Nikodym Theorem) the given identity $\mu(B) = \mu(aB)$ can be written in the form

$$\int_B f\,d\lambda = \int_{aB} f\,d\lambda.$$

For $B = [1, x]$ (put $B = [x, 1]$ if $0 < x < 1$), we get

$$\int_1^x f(t)\,dt = \int_a^{ax} f(t)\,dt$$

for each $a > 0$ and each $x > 0$. Differentiating with respect to x (and taking into account the Fundamental Theorem of Calculus), we see that

$$f(x) = af(ax)$$

holds for each $x > 0$ and each $a > 0$. Letting $x = 1$, we obtain

$$f(a) = \frac{f(1)}{a}$$

for all $a > 0$, as desired.

38. THE RIESZ REPRESENTATION THEOREM

Problem 38.1. *If X is a compact topological space, then show that a continuous linear functional F on $C(X)$ is positive if and only if $F(\mathbf{1}) = \|F\|$ holds.*

Solution. Let F be a continuous linear functional on $C(X)$, where X is compact. Note first that

$$\{f \in C(X)\colon \|f\|_\infty \leq 1\} = \{f \in C(X)\colon |f| \leq \mathbf{1}\}.$$

Thus, if F is also positive, then

$$\begin{aligned}\|F\| &= \sup\{F(f)\colon f \in C(X) \text{ and } \|f\|_\infty \leq 1\} \\ &= \sup\{F(f)\colon f \in C(X) \text{ and } |f| \leq \mathbf{1}\} = F(\mathbf{1}).\end{aligned}$$

On the other hand, assume $F(\mathbf{1}) = \|F\|$. Let $0 \leq f \in C(X)$ be nonzero, and put $g = \frac{f}{\|f\|_\infty}$. Clearly, $\|\mathbf{1} - g\|_\infty \leq 1$. Thus,

$$F(\mathbf{1}) - F(g) = F(\mathbf{1} - g) \leq \|F\| = F(\mathbf{1})$$

holds, which implies $F(g) \geq 0$. Therefore, $F(f) = \|f\|_\infty F(g) \geq 0$ holds, and so F is a positive linear functional.

Problem 38.2. *Let X be a compact topological space, and let F and G be two positive linear functionals on $C(X)$. If $F(\mathbf{1}) + G(\mathbf{1}) \leq \|F - G\|$, then show that $F \wedge G = 0$.*

Solution. Since $F, G \geq 0$, it follows that $F - G \leq F \vee G$ and $G - F \leq F \vee G$, and so $|F - G| \leq F \vee G$. Thus, by the preceding problem

$$\begin{aligned} \|F - G\| &\leq \|F \vee G\| = F \vee G(\mathbf{1}) \leq \|F + G\| \leq \|F\| + \|G\| \\ &= F(\mathbf{1}) + G(\mathbf{1}) \leq \|F - G\|, \end{aligned}$$

and hence, $F \vee G(\mathbf{1}) = F(\mathbf{1}) + G(\mathbf{1})$ holds. From $F + G = F \vee G + F \wedge G$, it follows that

$$\|F \wedge G\| = F \wedge G(\mathbf{1}) = F(\mathbf{1}) + G(\mathbf{1}) - F \vee G(\mathbf{1}) = 0,$$

and so $F \wedge G = 0$.

Problem 38.3. *Let X be a Hausdorff locally compact topological space and let*

$$c_0(X) = \{f \in C(X) \colon \forall \epsilon > 0 \, \exists K \text{ compact with } |f(x)| < \epsilon \, \forall x \notin K\}.$$

Show that:

a. *$c_0(X)$ equipped with the sup norm is a Banach lattice.*
b. *The norm completion of $C_c(X)$ is the Banach lattice $c_0(X)$.*

Solution. (a) Clearly, $c_0(X)$ with the sup norm is a normed vector lattice. For the completeness, let $\{f_n\}$ be a Cauchy sequence of $c_0(X)$. Then $\{f_n\}$ converges uniformly on X to some function f. By Theorem 9.2 we infer that $f \in C(X)$. Now, if $\varepsilon > 0$ is given, pick some n with $\|f_n - f\|_\infty < \varepsilon$, and then choose some compact set K with $|f_n(x)| < \varepsilon$ for $x \notin K$. Thus,

$$|f(x)| \leq |f_n(x) - f(x)| + |f_n(x)| < \varepsilon + \varepsilon = 2\varepsilon$$

holds for all $x \notin K$, so that $f \in c_0(X)$.

(b) Obviously, $C_c(X)$ is a vector sublattice of $c_0(X)$. We have to show that $C_c(X)$ is dense in $c_0(X)$.

To this end, let $f \in c_0(X)$ and let $\varepsilon > 0$. Pick some compact set K with $|f(x)| < \varepsilon$ for $x \notin K$, and then choose some open set V with compact closure such that $K \subseteq V$. By Theorem 10.8 there exists a function $g: X \longrightarrow \mathbf{R}$ with $K \prec g \prec V$. Then, $fg \in C_c(X)$ and $\|f - fg\|_\infty \leq 2\varepsilon$ holds, proving that $C_c(X)$ is dense in the Banach lattice $c_0(X)$.

Problem 38.4. *Let F be a positive linear functional on $C_c(X)$, where X is Hausdorff and locally compact, and let μ be the outer measure induced by F on X. Show that if μ^* is the outer measure generated by the measure space $(X, \mathcal{B}, \mu)$, then $\mu^*(A) = \mu(A)$ holds for every subset A of X.*

Solution. Let $A \subseteq X$. We know that

$$\mu(A) = \inf\{\mu(V) \colon V \text{ open and } A \subseteq V\}.$$

So, if $A \subseteq V$ holds with V open, then $\mu^*(A) \leq \mu^*(V) = \mu(V)$ also holds, and thus $\mu^*(A) \leq \mu(A)$. On the other hand, by Theorem 15.11 there exists some $B \in \mathcal{B}$ with $A \subseteq B$ and $\mu^*(A) = \mu(B)$. Thus, $\mu(A) \leq \mu(B) = \mu^*(A)$, proving that $\mu^*(A) = \mu(A)$ holds.

Problem 38.5. *Let μ and ν be two regular Borel measures on a Hausdorff locally compact topological space X. Then show that $\mu \geq \nu$ holds if and only if $\int f\,d\mu \geq \int f\,d\nu$ for each $f \in C_c(X)^+$.*

Solution. Let μ and ν be two regular Borel measures on a Hausdorff locally compact topological space X.

Assume first that $\mu \geq \nu$ holds (i.e., assume that $\mu(A) \geq \nu(A)$ holds for each $A \in \mathcal{B}$). Clearly, if ϕ is a μ-step function of the form $\phi = \sum_{i=1}^n a_i \chi_{A_i}$ with each $a_i \geq 0$ and each $A_i \in \mathcal{B}$, then ϕ is a ν-step function and $\int \phi\,d\mu \geq \int \phi\,d\nu$ holds. Now, let $0 \leq f \in C_c(X)$. Since

$$f^{-1}([a, b)) = \left[f^{-1}((-\infty, a))\right]^c \cap f^{-1}((-\infty, b)) \in \mathcal{B},$$

it follows from Theorem 17.7 that there exists a sequence $\{\phi_n\}$ of μ-step functions of the preceding type satisfying $\phi_n(x) \uparrow f(x)$ for each $x \in X$. This implies

$$\int f\,d\mu = \lim_{n\to\infty} \int \phi_n\,d\mu \geq \lim_{n\to\infty} \int \phi_n\,d\nu = \int f\,d\nu.$$

For the converse, assume that $\int f\,d\mu \geq \int f\,d\nu$ holds for each $0 \leq f \in C_c(X)$. In view of the regularity of the measures, in order to establish that $\mu \geq \nu$ holds it suffices to show that $\mu(K) \geq \nu(K)$ holds for each compact set K. To this end,

let K be a compact set. Given $\varepsilon > 0$, choose an open set V such that $K \subseteq V$ and $\mu(V) < \mu(K)+\varepsilon$. By Theorem 10.8 there exists a function $f \in C_c(X)$ such that $K \prec f \prec V$. Now note that from $\chi_K \le f \le \chi_V$, it follows that

$$\nu(K) = \int \chi_K \, d\nu \le \int f \, d\nu \le \int f \, d\mu \le \int \chi_V \, d\mu = \mu(V) < \mu(K) + \varepsilon$$

for all $\varepsilon > 0$. That is, $\nu(K) \le \mu(K)$ holds, as desired.

Problem 38.6. *Fix a point x in a Hausdorff locally compact topological space X, and define $F(f) = f(x)$ for each $f \in C_c(X)$. Show that F is a positive linear functional on $C_c(X)$ and then describe the unique regular Borel measure μ that satisfies $F(f) = \int f \, d\mu$ for each $f \in C_c(X)$. What is the support of μ?*

Solution. Clearly, F is a positive linear functional. The regular Borel measure representing F is the Dirac measure with "base point" at x. Its support is, of course, the set $\{x\}$.

Problem 38.7. *Let X be a compact Hausdorff topological space. If μ and ν are regular Borel measures, then show that the regular Borel measures $\mu \vee \nu$ and $\mu \wedge \nu$ satisfy*

a. $\operatorname{Supp}(\mu \vee \nu) = \operatorname{Supp}\mu \cup \operatorname{Supp}\nu$, *and*
b. $\operatorname{Supp}(\mu \wedge \nu) \subseteq \operatorname{Supp}\mu \cap \operatorname{Supp}\nu$.

Use (b) to show that if $\operatorname{Supp}\mu \cap \operatorname{Supp}\nu = \emptyset$, then $\mu \perp \nu$ holds. Also, give an example for which $\operatorname{Supp}(\mu \wedge \nu) \neq \operatorname{Supp}\mu \cap \operatorname{Supp}\nu$.

Solution. (a) Let $A = \operatorname{Supp}(\mu \vee \nu)$, $B = \operatorname{Supp}\mu$, and $C = \operatorname{Supp}\nu$. From $\mu \le \mu \vee \nu$, $\nu \le \mu \vee \nu$, and $\mu \vee \nu(A^c) = 0$, it follows that $\mu(A^c) = \nu(A^c) = 0$, and so $B \subseteq A$ and $C \subseteq A$. That is, $B \cup C \subseteq A$. On the other hand, the inequality $\mu \vee \nu \le \mu + \nu$ implies

$$\mu \vee \nu\left(B^c \cap C^c\right) \le (\mu + \nu)\left(B^c \cap C^c\right) \le \mu\left(B^c\right) + \nu\left(C^c\right) = 0,$$

and so $A \subseteq \left(B^c \cap C^c\right)^c = B \cup C$.
(b) The inclusion follows easily from the inequalities

$$\mu \wedge \nu \le \mu \quad \text{and} \quad \mu \wedge \nu \le \nu.$$

If $\operatorname{Supp}\mu \cap \operatorname{Supp}\nu = \emptyset$, then by part (b) $\operatorname{Supp}(\mu \wedge \nu) = \emptyset$ holds, and so $\mu \wedge \nu = 0$. For an example showing that equality need not hold in (b), let $X = \mathbf{R}$, $\mu =$ the Lebesgue measure, and $\nu =$ the Dirac measure with "base point" at 0.

By Problem 36.5, we have $\mu \wedge \nu = 0$. Therefore, $\operatorname{Supp}(\mu \wedge \nu) = \emptyset$ holds, while $\operatorname{Supp}\mu \cap \operatorname{Supp}\nu = \mathbb{R} \cap \{0\} = \{0\}$.

Problem 38.8. *Let X be a Hausdorff locally compact topological space X. Characterize the positive linear functionals F on $C_c(X)$ that are also lattice homomorphisms; that is, $F(f \vee g) = \max\{F(f), F(g)\}$ holds for each pair $f, g \in C_c(X)$.*

Solution. Let F be a positive linear functional on $C_c(X)$. Then we shall show that F is a lattice homomorphism if and only if there exist some $c \geq 0$ and some $a \in X$ such that $F(f) = cf(a)$ holds for all $f \in C_c(X)$.

Clearly, if for some $c \geq 0$ and some $a \in X$ we have $F(f) = cf(a)$ for each $f \in C_c(X)$, then F is a lattice homomorphism. For the converse, assume that F is a non-zero lattice homomorphism. Let μ be the regular Borel measure that represents F. If $x, y \in \operatorname{Supp}\mu$ satisfy $x \neq y$, then it is not difficult to see that there exist f, g in $C_c(X)$ with $f \wedge g = 0$ and $f(x) = g(y) = 1$. Therefore,

$$F(f \vee g) = F(f + g) = F(f) + F(g) > \max\{F(f),\ F(g)\}$$

must hold, which is a contradiction. Thus, $\operatorname{Supp}\mu$ consists precisely of one point; let $\operatorname{Supp}\mu = \{a\}$. Set $c = \mu(\{a\}) > 0$, and note that for every $f \in C_c(X)$, we have

$$F(f) = \int f\, d\mu = f(a) \cdot \mu(\{a\}) = cf(a).$$

Problem 38.9. *Let X be a Hausdorff locally compact topological space such that X is an uncountable set. Then show that*

a. *$C_c^*(X)$ is not separable, and*
b. *$C[0, 1]$ (with the sup norm) is not a reflexive Banach space.*

Solution. (a) For each $x \in X$ define the positive linear functional $F_x\colon C_c(X) \longrightarrow \mathbb{R}$ by $F_x(f) = f(x)$ and note that $\|F_x - F_y\| = 2$ holds for $x \neq y$. Clearly, the set $\{F_x\colon x \in X\}$ is an uncountable subset of $C_c^*(X)$. Therefore, $\{B(F_x, 1)\colon x \in X\}$ is an uncountable collection of pairwise disjoint open balls. From this, it easily follows that no countable subset of $C_c^*(X)$ can be dense in $C_c^*(X)$.
(b) By Problem 11.12, we know that $C[0, 1]$ is separable. If $C[0, 1]$ is reflexive, then its second dual is likewise separable. But then (by Problem 29.8) its first dual must be separable, contradicting part (a). Thus, $C[0, 1]$ is not a reflexive Banach lattice.

Problem 38.10. *Let X be a Hausdorff locally compact topological space. For a finite signed measure μ on $\mathcal{B}$ show that the following statements are equivalent:*

a. *μ belongs to $M_b(X)$.*
b. *μ^+ and μ^- are both finite regular Borel measures.*
c. *For each $A \in \mathcal{B}$ and $\epsilon > 0$, there exist a compact set K and an open set V with $K \subseteq A \subseteq V$ such that $|\mu(B)| < \epsilon$ holds for all $B \in \mathcal{B}$ with $B \subseteq V \setminus K$.*

Solution. (a)$\Longrightarrow$(b) Pick two finite regular Borel measures μ_1 and μ_2 such that $\mu = \mu_1 - \mu_2$. Then, $\mu^+ = (\mu_1 - \mu_2)^+ = \mu_1 \vee \mu_2 - \mu_2$ holds. By Theorem 38.5, $\mu_1 \vee \mu_2$ is a finite regular Borel measure, and from this it follows that μ^+ is a finite regular Borel measure. Similarly, μ^- is a finite regular Borel measure.
(b)$\Longrightarrow$(c) Note that $|\mu| = \mu^+ + \mu^-$ is a finite regular Borel measure. Now, let $A \in \mathcal{B}$ and let $\varepsilon > 0$ be given. Then, there exists a compact set K and an open set V with $K \subseteq A \subseteq V$ and $|\mu|(V \setminus K) < \varepsilon$. Therefore, if $B \in \mathcal{B}$ satisfies $B \subseteq V \setminus K$, then $|\mu(B)| \leq |\mu|(B) \leq |\mu|(V \setminus K) < \varepsilon$ holds.
(c)$\Longrightarrow$(a) Let $A \in \mathcal{B}$ and let $\varepsilon > 0$. Choose a compact set K and an open set V so that (c) is satisfied. Then, by Theorem 36.9, we have

$$0 \leq \mu^+(A) - \mu^+(K) = \mu^+(A \setminus K) = \sup\{\mu(B): B \in \mathcal{B} \text{ and } B \subseteq A \setminus K\}$$

and

$$0 \leq \mu^+(V) - \mu^+(A) = \mu^+(V \setminus A) = \sup\{\mu(B): B \in \mathcal{B} \text{ and } B \subseteq V \setminus A\}.$$

Thus, $\mu^+(A) - \mu^+(K) \leq \varepsilon$ and $\mu^+(V) - \mu^+(A) \leq \varepsilon$ both hold. Hence, μ^+ is a finite regular Borel measure. Similarly, μ^- is a finite regular Borel measure, and so $\mu = \mu^+ - \mu^- \in M_b(X)$.

Problem 38.11. *A sequence $\{x_n\}$ in a normed space is said to* **converge weakly** *to some vector x if* $\lim f(x_n) = f(x)$ *holds for every continuous linear functional f.*

a. *Show that a sequence in a normed space can have at most one weak limit.*
b. *Let X be a Hausdorff compact topological space. Then show that a sequence $\{f_n\}$ of $C(X)$ converges weakly to some function f if and only if $\{f_n\}$ is norm bounded and* $\lim f_n(x) = f(x)$ *holds for each $x \in X$.*

Solution. (a) Assume that a sequence $\{x_n\}$ in a normed vector space Y satisfies $\lim f(x_n) = f(x)$ and $\lim f(x_n) = f(y)$ for every $f \in Y^*$. Then, $f(x - y) = 0$ holds for all $f \in Y^*$. By Theorem 29.4, we see that $x - y = 0$, and so $\{x_n\}$ can have at most one weak limit.
(b) Assume first that the sequence $\{f_n\}$ of $C(X)$ converges weakly to some

function $f \in C(X)$. By Theorem 29.8, $\{f_n\}$ is norm bounded. If $x \in X$, let μ_x denote the Dirac measure with support $\{x\}$, and note that

$$f_n(x) = \int f_n \, d\mu_x \longrightarrow \int f \, d\mu_x = f(x).$$

Conversely, if $\{f_n\}$ is norm bounded and $\lim f_n(x) = f(x)$ holds for each $x \in X$ (where, of course, $f \in C(X)$), then the Lebesgue Dominated Convergence Theorem implies that $\lim \int f_n \, d\mu = \int f \, d\mu$ holds for every Borel measure μ. This, coupled with the Riesz Representation Theorem, shows that $\{f_n\}$ converges weakly to f.

Problem 38.12. *Let μ be a regular Borel measure on a Hausdorff locally compact topological space X, and let $f \in L_1(\mu)$. Show that the finite signed measure ν, defined by*

$$\nu(E) = \int_E f \, d\mu$$

for each Borel set E, is a (finite) regular Borel signed measure. In other words, show that $\nu \in M_b(X)$.

Solution. We can assume that $f(x) \geq 0$ holds for all x. By Problem 22.7, the set $A = \{x \in X: f(x) > 0\}$ is a σ-finite set with respect to μ. Choose a sequence $\{X_n\}$ of μ-measurable sets with $\mu(X_n) < \infty$ for each n and $X_n \uparrow A$.

Now, let E be a Borel set and let $\varepsilon > 0$; clearly, $\nu(E) = \nu(E \cap A)$. Select some n with $\nu(E) - \nu(X_n \cap E) < \varepsilon$. Also, using the regularity of μ and Problem 22.6, we see that there exists a compact set $K \subseteq X_n \cap E$ with

$$\nu(X_n \cap E) - \nu(K) = \int_{X_n \cap E} f \, d\mu - \int_K f \, d\mu < \varepsilon.$$

Thus, the compact set $K \subseteq E$ satisfies

$$0 \leq \nu(E) - \nu(K) = \big[\nu(E) - \nu(X_n \cap E)\big] + \big[\nu(X_n \cap E) - \nu(K)\big] < 2\varepsilon.$$

Next, use Problem 22.6 and the regularity of μ to see that for each n there exists an open set V_n satisfying $X_n \cap E \subseteq V_n$ and $\nu(V_n) - \nu(X_n \cap E) < \frac{\varepsilon}{2^n}$. Then, the open set $V = \bigcup_{n=1}^{\infty} V_n$ satisfies $E \subseteq V$, and in view of $V \setminus E \subseteq \bigcup_{n=1}^{\infty}(V_n \setminus X_n \cap E)$, we see that

$$0 \leq \nu(V) - \nu(E) = \nu(V \setminus E) \leq \sum_{n=1}^{\infty} \nu(V_n \setminus X_n \cap E) < \varepsilon.$$

Altogether, the preceding show that ν is a regular Borel measure.

Problem 38.13. *Generalize part (3) of Theorem 38.5 as follows: If μ and ν are two regular Borel measures on a Hausdorff locally compact topological space and one of then is σ-finite, then show that $\mu \wedge \nu$ is also a regular Borel measure.*

Solution. Let μ and ν be two regular Borel measures on a locally compact Hausdorff topological space X and assume that μ is σ-finite. Also, let $\omega = \mu \wedge \nu$ and note (in view of $\omega \le \mu$) that ω is a σ-finite Borel measure which is absolutely continuous with respect to μ.

Now, let E be a Borel subset of X satisfying $\mu(E) < \infty$ and let $\varepsilon > 0$. Consider ω and μ restricted to the Borel sets $\mathcal{B}_E$ of E (from Problem 12.13 we know that $\mathcal{B}_E = \{B \cap E \colon B \in \mathcal{B}\}$). Now, by the Radon–Nikodym Theorem there exists a (unique) non-negative function $f \in L_1(E, \mathcal{B}_E, \mu)$ satisfying

$$\omega(B \cap E) = \int_{B \cap E} f \, d\mu, \qquad \text{for each} \quad B \in \mathcal{B}.$$

Since μ is a regular Borel measure, it follows from Problem 22.6 that there exists a compact subset K of E such that

$$0 \le \omega(E) - \omega(K) = \int_E f \, d\mu - \int_K f \, d\mu = \int_{E \setminus K} f \, d\mu < \varepsilon.$$

Therefore, we infer that

$$\omega(E) = \sup\{\omega(K) \colon K \text{ compact and } K \subseteq E\}. \tag{$\star$}$$

Now, use the σ-finiteness of ω to show that ($\star$) holds true for each Borel subset of E.

It remains to be shown that the measure of every Borel set can be approximated from above by the measures of the open sets. To this end, let E be an arbitrary Borel set, and recall that

$$\omega(E) = \mu \wedge \nu(E) = \inf\{\mu(B) + \nu(E \setminus B) \colon B \in \mathcal{B} \text{ and } B \subseteq E\}.$$

Let $c = \inf\{\omega(O) \colon O \text{ open and } E \subseteq O\}$ and let $\varepsilon > 0$. Given $B \in \mathcal{B}$ with $B \subseteq E$, choose open sets V and W such that $B \subseteq V$, $E \setminus B \subseteq W$, $\mu(V) \le \mu(B) + \varepsilon$, and $\nu(W) \le \nu(E \setminus B) + \varepsilon$. Then, we have

$$\begin{aligned} \omega(E) &\le c \le \omega(V \cup W) \le \omega(V) + \omega(W) \le \mu(V) + \nu(W) \\ &\le \mu(B) + \varepsilon + \nu(E \setminus B) + \varepsilon = \mu(B) + \nu(E \setminus B) + 2\varepsilon. \end{aligned}$$

Thus, $\omega(E) \le c \le \omega(E) + 2\varepsilon$ holds for each $\varepsilon > 0$, and so $\omega(E) = c$, and we are finished.

Problem 38.14. *Show that every finite Borel measure on a complete separable metric space is a regular Borel measure. Use this conclusion to present an alternate proof of the fact that the Lebesgue measure is a regular Borel measure.*

Solution. Let X be a complete separable metric space, let $\mathcal{B}$ be the σ-algebra of all Borel sets of X, let $\{x_1, x_2, \dots\}$ be a dense countable subset of X, and let $\mu\colon \mathcal{B} \longrightarrow [0, \infty)$ be a measure.

Consider the collection $\mathcal{A}$ of subsets of X defined by

$$\begin{aligned}\mathcal{A} = \Big\{A \in \mathcal{B}\colon\ \mu(A) &= \inf\{\mu(O)\colon\ A \subseteq O \text{ and } O \text{ open}\}\\ &= \sup\{\mu(K)\colon\ K \subseteq A \text{ and } K \text{ compact}\}\Big\}.\end{aligned}$$

The collection $\mathcal{A}$ has the following properties:

1. $\mathcal{A}$ contains the open and closed sets.

To see this, assume first that V is an open set, and let $\varepsilon > 0$. For each n let $\mathcal{F}_n$ be the collection of all open balls of the form $B(x_i, r)$ with r a rational number less than or equal to $\frac{1}{n}$ and $\overline{B(x_i, r)} \subseteq V$. Clearly, each $\mathcal{F}_n$ is at most countable and $V = \bigcup_{B\in\mathcal{F}_n} B$ holds. For each n pick $B_1^n, \dots, B_{k_n}^n \in \mathcal{F}_n$ such that

$$\mu\Big(V \setminus \bigcup_{i=1}^{k_n} B_i^n\Big) < \frac{\varepsilon}{2^n}.$$

Next, put $C = \bigcap_{n=1}^{\infty} \bigcup_{i=1}^{k_n} B_i^n$, and note that C is a totally bounded set. Hence, its closure $\overline{C}$ is also a totally bounded set (why?). Since (by Theorem 6.13) $\overline{C}$ is a complete metric space in its own right, it follows from Theorem 7.8 that $\overline{C}$ is a compact set. Now, note that $\overline{C} \subseteq V$ holds, and that

$$\begin{aligned}0 \le \mu(V) - \mu(\overline{C}) &\le \mu(V) - \mu(C) = \mu(V \setminus C)\\ &= \mu\Big(\bigcup_{n=1}^{\infty}\Big(V \setminus \bigcup_{i=1}^{k_n} B_i^n\Big)\Big) \le \sum_{n=1}^{\infty} \mu\Big(V \setminus \bigcup_{i=1}^{k_n} B_i^n\Big)\\ &< \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon.\end{aligned}$$

Therefore,

$$\begin{aligned}\mu(V) &= \inf\{\mu(O)\colon\ V \subseteq O \text{ and } O \text{ open}\}\\ &= \sup\{\mu(K)\colon\ K \subseteq V \text{ and } K \text{ compact}\}\end{aligned}$$

holds, and so $V \in \mathcal{A}$.

Now, let C be a closed set, and let $\varepsilon > 0$. By the preceding, there exists a compact subset K with $\mu(X) - \mu(K) < \varepsilon$. Then, the compact subset $C \cap K$ of C satisfies

$$\begin{aligned}\mu(C) - \mu(C \cap K) &= \mu(C \setminus C \cap K)\\ &= \mu(C \setminus K) \leq \mu(X \setminus K) = \mu(X) - \mu(K) < \varepsilon.\end{aligned}$$

Also, by the previous part, there exists a compact set K_1 with $K_1 \subseteq X \setminus C$ and $\mu(X \setminus C) - \mu(K_1) < \varepsilon$. Now, the open set $O = X \setminus K_1$ satisfies $C \subseteq O$ and

$$\begin{aligned}\mu(O) - \mu(C) &= \mu(X \setminus K_1) - \mu(C) = \mu(X) - \mu(K_1) - \mu(C)\\ &= \mu(X \setminus C) - \mu(K_1) < \varepsilon.\end{aligned}$$

Thus,

$$\begin{aligned}\mu(C) &= \inf\{\mu(O)\colon\ C \subseteq O \text{ and } O \text{ open}\}\\ &= \sup\{\mu(K)\colon\ K \subseteq C \text{ and } K \text{ compact}\}\end{aligned}$$

also holds, and so $C \in \mathcal{A}$.

2. If $A \in \mathcal{A}$, then $A^c \in \mathcal{A}$.

From $\mu(A) = \sup\{\mu(K)\colon\ K \subseteq A \text{ and } K \text{ compact}\}$, it follows that

$$\begin{aligned}\mu(A^c) &= \mu(X) - \mu(A)\\ &= \inf\{\mu(X) - \mu(K)\colon\ K \subseteq A \text{ and } K \text{ compact}\}\\ &= \inf\{\mu(K^c)\colon\ K \subseteq A \text{ and } K \text{ compact}\}\\ &= \inf\{\mu(O)\colon\ A^c \subseteq O \text{ and } O \text{ open}\}.\end{aligned}$$

Similarly, $\mu(A) = \inf\{\mu(O)\colon\ A \subseteq O \text{ and } O \text{ open}\}$ implies

$$\mu(A^c) = \sup\{\mu(C)\colon\ C \subseteq A^c \text{ and } C \text{ closed}\}.$$

Since, by part (1), $\mu(C) = \sup\{\mu(K)\colon\ K \subseteq C \text{ and } K \text{ compact}\}$ holds for each closed set C, we see that

$$\mu(A^c) = \sup\{\mu(K)\colon\ K \subseteq A^c \text{ and } K \text{ compact}\}.$$

3. If $\{A_n\}$ is a sequence of $\mathcal{A}$, then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$.

Let $\{A_n\} \subseteq \mathcal{A}$, let $A = \bigcup_{n=1}^{\infty} A_n$, and let $\varepsilon > 0$. For each n pick some open set O_n with $A_n \subseteq O_n$ and $\mu(O_n \setminus A_n) < \varepsilon 2^{-n}$. Then, the open set $O = \bigcup_{n=1}^{\infty} O_n$

satisfies $A \subseteq O$ and from $O \setminus A = \bigcup_{n=1}^{\infty} O_n \setminus \bigcup_{n=1}^{\infty} A_n \subseteq \bigcup_{n=1}^{\infty}(O_n \setminus A_n)$, we get

$$\mu(O) - \mu(A) = \mu(O \setminus A) \le \mu\Big(\bigcup_{n=1}^{\infty}(O_n \setminus A_n)\Big) \le \sum_{n=1}^{\infty} \mu(O_n \setminus A_n) < \varepsilon.$$

On the other hand, fix some k with $\mu(A \setminus \bigcup_{i=1}^{k} A_i) < \varepsilon$, and then for each $1 \le i \le k$ pick a compact set $K_i \subseteq A_i$ with $\mu(A_i \setminus K_i) < \varepsilon 2^{-i}$. Then, the compact set $K = \bigcup_{i=1}^{k} K_i$ satisfies $K \subseteq \bigcup_{i=1}^{k} A_i \subseteq A$ and

$$\begin{aligned}
\mu(A) - \mu(K) &= \mu(A \setminus K) = \mu\Big(A \setminus \bigcup_{i=1}^{k} A_i\Big) + \mu\Big(\Big(\bigcup_{i=1}^{k} A_i\Big) \setminus K\Big) \\
&< \varepsilon + \mu\Big(\Big(\bigcup_{i=1}^{k} A_i\Big) \setminus K\Big) = \varepsilon + \mu\Big(\bigcup_{i=1}^{k} A_i \setminus \bigcup_{i=1}^{k} K_i\Big) \\
&\le \varepsilon + \sum_{i=1}^{k} \mu(A_i \setminus K_i) < \varepsilon + \varepsilon = 2\varepsilon.
\end{aligned}$$

The validity of statement (3) has been established.

Now, from the preceding statements, we see that $\mathcal{A}$ is a σ-algebra that contains the open sets. Consequently, every Borel set belongs to $\mathcal{A}$ (i.e., $\mathcal{A} = \mathcal{B}$), and so μ is a regular Borel measure.

Now, let us use the previous conclusion to establish that the Lebesgue measure λ on $\mathbf{R}^n$ is a regular Borel measure. To this end, let A be an arbitrary Borel set. Also, let V_n (resp. C_n) denote the open (resp. the closed) ball of $\mathbf{R}^n$ with center at zero and radius n. Since each C_n is a complete separable metric space in its own right, it follows from the previous result that λ restricted to each C_n is a regular Borel measure. Therefore, we have

$$\lambda(A \cap C_n) = \sup\{\lambda(K)\colon\ K \subseteq A \cap C_n \text{ and } K \text{ compact}\}.$$

From $A \cap C_n \uparrow A$, it follows that $\lambda(A \cap C_n) \uparrow \lambda(A)$, and an easy argument shows that

$$\lambda(A) = \sup\{\lambda(K)\colon\ K \subseteq A \text{ and } K \text{ compact}\}.$$

Next, note that if $\lambda(A) = \infty$, then

$$\lambda(A) = \inf\{\lambda(O)\colon\ A \subseteq O \text{ and } O \text{ open}\}$$

is trivially true. So, assume that $\lambda(A) < \infty$, and let $\varepsilon > 0$. By the regularity of λ on C_n (and the fact that V_n is an open set), we see that

$$\begin{aligned}
\lambda(A \cap V_n) &= \inf\{\lambda(O \cap C_n)\colon\ V_n \cap A \subseteq O \cap C_n \text{ and } O \text{ open}\} \\
&= \inf\{\lambda(O \cap V_n)\colon\ V_n \cap A \subseteq O \cap V_n \text{ and } O \text{ open}\} \\
&= \inf\{\lambda(O)\colon\ A \cap V_n \subseteq O \text{ and } O \text{ open}\}.
\end{aligned}$$

Therefore, for each n there exists an open set O_n with $A \cap V_n \subseteq O_n$ and $\lambda(O_n \setminus A \cap V_n) < \varepsilon 2^{-n}$. Now, the set $O = \bigcup_{n=1}^{\infty} O_n$ is open and satisfies $A \subseteq O$. From

$$O \setminus A = \bigcup_{n=1}^{\infty} O_n \setminus \bigcup_{n=1}^{\infty} A \cap V_n \subseteq \bigcup_{n=1}^{\infty} (O_n \setminus A \cap V_n),$$

we see that

$$0 \leq \lambda(O) - \lambda(A) = \lambda(O \setminus A) \leq \sum_{n=1}^{\infty} \lambda(O_n \setminus A \cap V_n) < \varepsilon.$$

Hence, $\lambda(A) = \inf\{\lambda(O)\colon\ A \subseteq O \text{ and } O \text{ open}\}$ also holds, and so the Lebesgue measure λ is a regular Borel measure.

Problem 38.15. *Let X be a Hausdorff compact topological space. If $\phi\colon X \to X$ is a continuous function, then show that there exists a regular Borel measure on X such that*

$$\int f \circ \phi \, d\mu = \int f \, d\mu$$

holds for each $f \in C(X)$.

Solution. Let X be a Hausdorff compact topological space and let $\phi\colon X \to X$ be a continuous function. Fix some $\omega \in X$ and let $\mathcal{L}im\colon \ell_\infty \to \ell_\infty$ is a Banach–Mazur limit (see Problem 29.7). Now, consider the positive linear functional $F\colon C(X) \to \mathbb{R}$ defined by

$$F(f) = \mathcal{L}im(f(\phi(\omega)), f(\phi^2(\omega)), f(\phi^3(\omega)), \ldots),$$

and let μ be the regular Borel measure on X representing F, i.e., $F(f) = \int f \, d\mu$

holds for each $f \in C(X)$. The identity

$$\mathcal{L}im(x_1, x_2, \ldots) = \mathcal{L}im(x_2, x_3, \ldots)$$

for all $(x_1, x_2, \ldots) \in \ell_\infty$ easily implies $F(f) = F(f \circ \phi)$ for each $f \in C(X)$. Consequently, the regular Borel measure μ satisfies $\int f\, d\mu = \int f \circ \phi\, d\mu$ for each $f \in C(X)$.

Problem 38.16. *This exercise gives an identification of the order dual $C_c^\sim(X)$ of $C_c(X)$. Consider the collection $\mathcal{M}(X)$ of all formal expressions $\mu_1 - \mu_2$ with μ_1 and μ_2 regular Borel measures. That is,*

$$\mathcal{M}(X) = \{\mu_1 - \mu_2 \colon\ \mu_1 \textit{ and } \mu_2 \textit{ are regular Borel measures on } X\}.$$

a. *Define $\mu_1 - \mu_2 \equiv \nu_1 - \nu_2$ in $\mathcal{M}(X)$ to mean $\mu_1(A) + \nu_2(A) = \nu_1(A) + \mu_2(A)$ for all $A \in \mathcal{B}$. Show that $\equiv$ is an equivalence relation.*
b. *Denote the collection of all equivalence classes by $\mathcal{M}(X)$ again. That is, $\mu_1 - \mu_2$ and $\nu_1 - \nu_2$ are considered to be identical if $\mu_1 + \nu_2 = \nu_1 + \mu_2$ holds. In $\mathcal{M}(X)$ define the algebraic operations*

$$(\mu_1 - \mu_2) + (\nu_1 - \nu_2) = (\mu_1 + \nu_1) - (\mu_2 + \nu_2),$$
$$\alpha(\mu_1 - \mu_2) = \begin{cases} \alpha\mu_1 - \alpha\mu_2 & \textit{if } \alpha \geq 0 \\ (-\alpha)\mu_2 - (-\alpha)\mu_1 & \textit{if } \alpha < 0. \end{cases}$$

Show that these operations are well defined (i.e., show that they depend only upon the equivalence classes) and that they make $\mathcal{M}(X)$ a vector space.
c. *Define an ordering in $\mathcal{M}(X)$ by $\mu_1 - \mu_2 \geq \nu_1 - \nu_2$ whenever*

$$\mu_1(A) + \nu_2(A) \geq \nu_1(A) + \mu_2(A)$$

holds for each $A \in \mathcal{B}$. Show that $\geq$ is well defined and that it is an order relation on $\mathcal{M}(X)$ under which $\mathcal{M}(X)$ is a vector lattice.
d. *Consider the mapping $\mu = \mu_1 - \mu_2 \mapsto F_\mu$ from $\mathcal{M}(X)$ to $C_c^\sim(X)$ defined by $F_\mu(f) = \int f\, d\mu_1 - \int f\, d\mu_2$ for each $f \in C_c(X)$. Show that F_μ is well defined and that $\mu \mapsto F_\mu$ is a lattice isomorphism (Lemma 38.6 may be helpful here) from $\mathcal{M}(X)$ onto $C_c^\sim(X)$. That is, show that $C_c^\sim(X) = \mathcal{M}(X)$ holds.*

Solution. (a) Clearly, $\mu_1 - \mu_2 \equiv \mu_1 - \mu_2$ and $\mu_1 - \mu_2 \equiv \nu_1 - \nu_2$ implies $\nu_1 - \nu_2 \equiv \mu_1 - \mu_2$. For the transitivity, let $\mu_1 - \mu_2 \equiv \nu_1 - \nu_2$ and $\nu_1 - \nu_2 \equiv \omega_1 - \omega_2$. That is, assume that $\mu_1 + \nu_2 = \nu_1 + \mu_2$ and $\nu_1 + \omega_2 = \omega_1 + \nu_2$. Adding the last two equalities, we see that

$$\mu_1 + \omega_2 + (\nu_1 + \nu_2) = \omega_1 + \mu_2 + (\nu_1 + \nu_2). \qquad (\star)$$

Since all measures involved are regular Borel measures, it follows from $(\star)$ that $\mu_1(K)+\omega_2(K)=\omega_1(K)+\mu_2(K)$ holds for each compact subset K of X. The regularity of the measures implies

$$\mu_1(A)+\omega_2(A)=\omega_1(A)+\mu_2(A)$$

for each $A \in \mathcal{B}$, and so $\mu_1-\mu_2=\omega_1-\omega_2$ holds.
(b) To see that the addition is well defined, assume that $\mu_1-\mu_2 \equiv \nu_1-\nu_2$ and $\omega_1-\omega_2 \equiv \pi_1-\pi_2$. That is, $\mu_1+\nu_2=\nu_1+\mu_2$ and $\omega_1+\pi_2=\pi_1+\omega_2$, and so $(\mu_1+\omega_1)+(\nu_2+\pi_2)=(\nu_1+\pi_1)+(\mu_2+\omega_2)$. That is,

$$\begin{aligned}(\mu_1-\mu_2)+(\omega_1-\omega_2)&=(\mu_1+\omega_1)-(\mu_2+\omega_2)\\ &\equiv(\nu_1+\pi_1)-(\nu_2+\pi_2)=(\nu_1-\nu_2)+(\pi_1-\pi_2).\end{aligned}$$

Similarly, the multiplication is well defined. Now, it is a routine matter to verify that under these algebraic operations $\mathcal{M}(X)$ is a vector space.
(c) To verify that $\geq$ is well defined, proceed as in part (b) above. It is a routine matter to check that $\geq$ makes $\mathcal{M}(X)$ a partially ordered vector space.

Next, we shall show that $\mathcal{M}(X)$ is a vector lattice. It suffices to verify that $(\mu_1-\mu_2)^+$ exists in $\mathcal{M}(X)$ for each $\mu_1-\mu_2 \in \mathcal{M}(X)$. To this end, let $\mu_1-\mu_2$ in $\mathcal{M}(X)$. By Theorem 38.5, $\mu_1 \vee \mu_2$ is a regular Borel measure, and we claim that $(\mu_1-\mu_2)^+=\mu_1\vee\mu_2-\mu_2$ holds in $\mathcal{M}(X)$. Clearly,

$$\mu_1-\mu_2 \leq \mu_1\vee\mu_2-\mu_2 \quad \text{and} \quad 0 \leq \mu_1\vee\mu_2-\mu_2$$

both hold. To see that $\mu_1\vee\mu_2-\mu_2$ is the least upper bound of $\mu_1-\mu_2$ and 0, assume $\mu_1-\mu_2 \leq \nu_1-\nu_2=\nu$ and $\nu \geq 0$. Then, $\nu+\mu_2$ is a regular Borel measure such that $\nu+\mu_2 \geq \mu_1$ and $\nu+\mu_2 \geq \mu_2$ both hold. By Theorem 38.5, $\nu+\mu_2 \geq \mu_1\vee\mu_2$, and hence $\nu \geq \mu_1\vee\mu_2-\mu_2$ holds in $\mathcal{M}(X)$. This shows that $\mu_1\vee\mu_2-\mu_2$ is the least upper bound of $\mu_1-\mu_2$ and 0.
(d) It is a routine matter to verify that $\mu \longmapsto F_\mu$ from $\mathcal{M}(X)$ into $C_c^{\sim}(X)$ is well defined and linear. Moreover, since every $F \in C_c^{\sim}(X)$ can be written as a difference of two positive linear functionals, the Riesz Representation Theorem guarantees that $\mu \longmapsto F_\mu$ is onto. To see that $\mu \longmapsto F_\mu$ is one-to-one, assume that $F_\mu=0$. Then, $\int f\,d\mu_1=\int f\,d\mu_2$ holds for each $f \in C_c(X)$, and so by (the Riesz Representation Theorem) $\mu_1=\mu_2$. Therefore, $\mu=\mu_1-\mu_2=0$ and so $\mu \longmapsto F_\mu$ is one-to-one.

Finally, observe that $\mu \geq 0$ holds in $\mathcal{M}(X)$ if and only if $F_\mu \geq 0$ holds in $C_c^{\sim}(X)$, and then invoke Lemma 38.6 to see that $\mu \longmapsto F_\mu$ is a lattice isomorphism from $\mathcal{M}(X)$ onto $C_c^{\sim}(X)$. Thus, under this lattice isomorphism, we can say that $C_c^{\sim}(X)=\mathcal{M}(X)$.

Problem 38.17. *This problem shows that for a noncompact space X, in general $C_c^*(X)$ is a proper ideal of $C_c^\sim(X)$. Let X be a Hausdorff locally compact topological space having a sequence $\{\mathcal{O}_n\}$ of open sets such that $\mathcal{O}_n \subseteq \mathcal{O}_{n+1}$ and $\mathcal{O}_n \neq \mathcal{O}_{n+1}$ for each n, and with $X = \bigcup_{n=1}^{\infty} \mathcal{O}_n$.*

a. *Show that if X is σ-compact but not a compact space, then X admits a sequence $\{\mathcal{O}_n\}$ of open sets with the preceding properties.*

b. *Choose $x_1 \in \mathcal{O}_1$ and $x_n \in \mathcal{O}_n \setminus \mathcal{O}_{n-1}$ for $n \geq 2$. Then, show that*

$$F(f) = \sum_{n=1}^{\infty} f(x_n) \quad for \quad f \in C_c(X)$$

defines a positive linear functional on $C_c(X)$ that is not continuous.

c. *Determine the (unique) regular Borel measure μ on X that represents F. What is the support of μ?*

Solution. (a) Let $\{K_n\}$ be a sequence of compact sets with $K_n \uparrow X$. For each n pick an open set V_n with compact closure such that $K_n \subseteq V_n$. Put $O_n = \bigcup_{i=1}^{n} V_i$ and note that $O_n \uparrow X$. Since each O_n has compact closure and X is not compact, $O_n \neq X$ holds for each n. By passing to a subsequence of $\{O_n\}$, we can assume that $O_n \neq O_{n+1}$ also holds for each n.

(b) Let $f \in C_c(X)$. Since $\operatorname{Supp} f \subseteq \bigcup_{n=1}^{\infty} O_n$ holds and $\operatorname{Supp} f$ is compact, there exists some k with $\operatorname{Supp} f \subseteq O_k$. Thus, $f(x_n) = 0$ for $n > k$, and so F clearly defines a positive linear functional on $C_c(X)$.

Next, we shall show that F is not continuous. By Theorem 10.8 there exists some $g_n \in C_c(X)$ with $\{x_1, \ldots, x_n\} \prec g_n \prec O_n$. Therefore, $\|F\| \geq F(g_n) = n$ holds for each n, and so $\|F\| = \infty$.

(c) The regular Borel measure μ that represents F is defined on the Borel set B by

$$\mu(B) = \text{The number of elements of } \{x_1, x_2, \ldots\} \cap B.$$

(If $\{x_1, x_2, \ldots\} \cap B$ is countable, then $\mu(B) = \infty$, and if $\{x_1, x_2, \ldots\} \cap B = \emptyset$, then $\mu(B) = 0$.) Also, note that

$$\operatorname{Supp} \mu = \{x_1, x_2, \ldots\}.$$

39. DIFFERENTIATION AND INTEGRATION

Problem 39.1. *If μ is a Borel measure on $\mathbf{R}^k$, then show that $\mu \perp \lambda$ holds if and only if $D\mu(x) = 0$ for almost all x.*

Solution. Assume $\mu \perp \lambda$. Choose two disjoint Borel sets A and B with $A \cup B = \mathbf{R}^k$ and $\mu(A) = \lambda(B) = 0$. By Lemma 39.3, $D\mu(x) = 0$ holds for almost all x in A, and so $D\mu(x) = 0$ holds for almost all x in $\mathbf{R}^k$.

Now, suppose that $D\mu(x) = 0$ holds for almost all $x \in \mathbf{R}^k$. Use the Lebesgue Decomposition Theorem 37.7 to write $\mu = \mu_1 + \mu_2$ with $\mu_1 \ll \lambda$ and $\mu_2 \perp \lambda$. By the preceding, $D\mu_2(x) = 0$ holds for almost all x in $\mathbf{R}^k$. Thus, from Theorem 39.4, it follows that

$$\tfrac{d\mu_1}{d\lambda} = D\mu_1 = D\mu = 0,$$

and so $\mu_1 = 0$. Therefore, $\mu = \mu_2 \perp \lambda$ holds.

Problem 39.2. *Show that if E is a Lebesgue measurable subset of $\mathbf{R}^k$, then almost all points of E are density points.*

Solution. For each $x = (x_1, \dots, x_k) \in \mathbf{R}^k$ and each $\varepsilon > 0$, consider the open interval $I_\varepsilon = \prod_{i=1}^k (x_i - \varepsilon, x_i + \varepsilon)$. If E is a Lebesgue measurable set, then

$$\lim_{\varepsilon \to 0^+} \tfrac{\lambda(E \cap I_\varepsilon)}{(2\varepsilon)^k} = \chi_E(x) \qquad (\star)$$

holds for almost all x. To see this, note first that we can assume without loss of generality that $\lambda(E) < \infty$ holds (why?). Now, consider the finite Borel measure μ on $\mathbf{R}^k$ defined by

$$\mu(A) = \lambda(E \cap A) = \int_A \chi_E \, d\lambda.$$

Clearly, $\mu \ll \lambda$ and $\frac{d\mu}{d\lambda} = \chi_E$. By Theorem 39.4, we have $D\mu = \chi_E$ a.e., and the validity of $(\star)$ follows.

Problem 39.3. *Write $B_r(a)$ for the open ball with center at $a \in \mathbf{R}^k$ and radius r. If f is a Lebesgue integrable function on $\mathbf{R}^k$, then a point $a \in \mathbf{R}^k$ is called a* **Lebesgue point** *for f if*

$$\lim_{r \to 0^+} \frac{1}{\lambda(B_r(a))} \int_{B_r(a)} |f(x) - f(a)| \, d\lambda(x) = 0.$$

Show that if f is a Lebesgue integrable function on $\mathbf{R}^k$, then almost all points of $\mathbf{R}^k$ are Lebesgue points.

Solution. Denote by Q the set of all rational numbers of $\mathbf{R}$. Fix some $a \in Q$, and let $B_n = B(0, n)$. Now, define the finite Borel measure μ by

$$\mu(E) = \int_{E \cap B_n} \big|f(x) - a\big|\, d\lambda(x).$$

Since $\mu \ll \lambda$, it follows from Theorem 39.4 that

$$D\mu = \tfrac{d\mu}{d\lambda} = |f - a|\chi_{B_n} \text{ a.e.}$$

Consequently,

$$\lim_{r \to 0^+} \tfrac{1}{\lambda(B_r(x))} \int_{B_r(x)} \big|f(t) - a\big|\, d\lambda(t) = \big|f(x) - a\big| \qquad (\star)$$

holds for almost all x in B_n, and therefore (since n is arbitrary) $(\star)$ holds for almost all x in $\mathbf{R}^k$. Let E_a be a Lebesgue null set for which $(\star)$ holds for all $x \notin E_a$. Set $E = \bigcup_{a \in Q} E_a$, and note that $\lambda(E) = 0$.

Now, let $y \notin E$ and let $\varepsilon > 0$. Choose some rational number $s \in Q$ with $|s - f(y)| < \varepsilon$ (we shall assume that f is real-valued everywhere). In view of $|f(x) - f(y)| \le |f(x) - s| + |s - f(y)|$, we see that

$$\begin{aligned}
&\limsup_{r \to 0^+} \tfrac{1}{\lambda(B_r(y))} \int_{B_r(y)} \big|f(x) - f(y)\big|\, d\lambda(x) \\
&\le \lim_{r \to 0^+} \tfrac{1}{\lambda(B_r(y))} \int_{B_r(y)} \big|f(x) - s\big|\, d\lambda(x) \\
&\quad + \lim_{r \to 0^+} \tfrac{1}{\lambda(B_r(y))} \int_{B_r(y)} \big|s - f(y)\big|\, d\lambda(x) \\
&= \big|f(y) - s\big| + \big|s - f(y)\big| < 2\varepsilon,
\end{aligned}$$

and from this the desired conclusion follows.

Problem 39.4. *Let $f\colon \mathbf{R} \to \mathbf{R}$ be an increasing, left continuous function. Show directly (i.e., without using Theorem 38.4) that the Lebesgue–Stieltjes measure μ_f is a regular Borel measure.*

Solution. Let (a, b) be an open interval. Then, there exists a sequence $\{[a_n, b_n]\}$ of closed intervals with $[a_n, b_n] \uparrow (a, b)$. It follows that $\mu_f([a_n, b_n]) \uparrow \mu_f((a, b))$. Since every open subset of $\mathbf{R}$ can be written as an at most countable union of pairwise disjoint open intervals, it follows that

$$\mu_f(O) = \sup\big\{\mu_f(K)\colon K \subseteq O \text{ and } K \text{ compact}\big\}$$

holds for all open sets O.

Now, let $[a, b)$ be a finite interval. Then, for each point $c < a$ of continuity of f, we have $[a, b) \subseteq (c, b)$ and $\mu_f\big((c, b)\big) - \mu_f\big([a, b)\big) = f(a) - f(c)$. By the left continuity of f, we see that $\mu_f([a, b)) = \inf\{\mu_f\big((c, b)\big)\colon c < a\}$. Next, consider a σ-set A with $\mu_f(A) < \infty$. Choose a pairwise disjoint sequence $\{[a_n, b_n)\}$ with $A = \bigcup_{n=1}^{\infty}[a_n, b_n)$. Given $\varepsilon > 0$, for each n choose some real number $c_n < a_n$ with $\mu_f\big((c_n, b_n) \setminus [a_n, b_n)\big) < \frac{\varepsilon}{2^n}$, and then set $V = \bigcup_{n=1}^{\infty}(c_n, b_n)$. Clearly, V is an open set, $A \subseteq V$, and

$$\mu_f(V) - \mu_f(A) = \mu_f(V \setminus A) \le \mu_f\Big(\bigcup_{n=1}^{\infty}\big[(c_n, b_n) \setminus [a_n, b_n)\big]\Big)$$
$$\le \sum_{n=1}^{\infty} \mu_f\big((c_n, b_n) \setminus [a_n, b_n)\big) < \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon.$$

Thus, $\mu_f(A) = \inf\{\mu_f(V)\colon A \subseteq V \text{ and } V \text{ open}\}$.

Now, to complete the proof, use Problem 15.2. (For a general result about regular Borel measures, see also Problem 38.14.)

Problem 39.5 (Fubini). *Let $\{f_n\}$ be a sequence of increasing functions defined on $[a, b]$ such that $\sum_{n=1}^{\infty} f_n(x) = f(x)$ converges in $\mathbf{R}$ for each $x \in [a, b]$. Then, show that f is differentiable almost everywhere and that $f'(x) = \sum_{n=1}^{\infty} f_n'(x)$ holds for almost all x.*

Solution. Replacing each f_n by $f_n - f_n(a)$, we can assume that $f_n \ge 0$ holds for each n. Set $s_n = f_1 + \cdots + f_n$, and note that each s_n is increasing and $s_n(x) \uparrow f(x)$ holds for each x. Clearly, f is also an increasing function. By Theorem 39.9, f and all the f_n are differentiable almost everywhere. Since $f_{n+1} = s_{n+1} - s_n$ is an increasing function, we see that $s_{n+1}'(x) \ge s_n'(x)$ must hold for almost all x. Similarly, since $f(x) - s_n(x) = \sum_{i=n+1}^{\infty} s_i(x)$ is an increasing function, it follows that $f'(x) \ge s_n'(x)$ holds for almost all x. Thus,

$$\lim_{n\to\infty} s_n'(x) = \sum_{n=1}^{\infty} f_n'(x)$$

exists for almost all x.

Now, for each n let

$$t_n(x) = f(x) - s_n(x) = \sum_{i=n+1}^{\infty} f_i(x) \ge 0.$$

Clearly, each t_n is an increasing function. Pick a subsequence $\{s_{k_n}\}$ of $\{s_n\}$ such that

$$\sum_{n=1}^{\infty} t_{k_n}(x) \leq \sum_{n=1}^{\infty} [f(b) - s_{k_n}(b)] < \infty.$$

The same arguments applied to $\{t_{k_n}\}$ instead of $\{s_n\}$ show that

$$\sum_{n=1}^{\infty} t'_{k_n}(x) = \sum_{n=1}^{\infty} [f'(x) - s'_{k_n}(x)]$$

converges for almost all x. In particular, $s'_{k_n}(x) \longrightarrow f'(x)$ holds for almost all x, and so

$$\sum_{n=1}^{\infty} f'_n(x) = f'(x)$$

holds for almost all x.

Problem 39.6. *Suppose $\{f_n\}$ is a sequence of increasing functions on $[a, b]$ and that f is an increasing function on $[a, b]$ such that $\mu_{f_n} \uparrow \mu_f$. Establish that $f'(x) = \lim f'_n(x)$ holds for almost all x.*

Solution. We shall present a solution of this problem based upon the following general continuity property of the Differential Operator D: *If $\{\mu_n\}$ is a sequence of Borel measures in $\mathbf{R}^k$ and $\mu_n \uparrow \mu$ holds for some Borel measure μ, then $D\mu_n \uparrow D\mu$ a.e. also holds.*

If this property is established, then using Theorem 39.8, we see that

$$f'_n(x) = D\mu_{f_n}(x) \uparrow D\mu_f(x) = f'(x)$$

must hold for almost all x.

To establish the validity of the continuity property start by observing that if two Borel measures μ and ν satisfy $\mu \leq \nu$, then $D\mu \leq D\nu$ a.e. holds. Indeed, by Theorem 39.6 both μ and ν are differentiable almost everywhere. If $x \in \mathbf{R}^k$ is a point for which $D\mu(x)$ and $D\nu(x)$ exist and $B_n = B(x, \frac{1}{n})$, then

$$D\mu(x) = \lim_{n\to\infty} \tfrac{\mu(B_n)}{\lambda(B_n)} \leq \lim_{n\to\infty} \tfrac{\nu(B_n)}{\lambda(B_n)} = D\nu(x).$$

Now, let $\mu_n \uparrow \mu$. Restricting ourselves to the open balls $\{x \in \mathbf{R}^k\colon \|x\| < n\}$, we can assume without loss of generality that all measures are finite.

By the Lebesgue Decomposition Theorem 37.7, we can write $\mu_n = \nu_n + \omega_n$ with $\nu_n \ll \lambda$ and $\omega_n \perp \lambda$. It follows from the proof of Theorem 37.7 that $\mu_n \wedge m\lambda \uparrow_m \nu_n$. Clearly, this implies $\nu_n \le \nu_{n+1}$ for each n. From formula (c) of Problem 9.1, we get

$$\mu_n - \mu_n \wedge m\lambda = 0 \vee (\mu_n - m\lambda) \le 0 \vee (\mu_{n+1} - m\lambda) = \mu_{n+1} - \mu_{n+1} \wedge m\lambda$$

for each m. Letting $m \longrightarrow \infty$, we obtain

$$\omega_n = \mu_n - \nu_n \le \mu_{n+1} - \nu_{n+1} = \omega_{n+1}$$

for each n. Let $\nu_n \uparrow \nu$ and $\omega_n \uparrow \omega$. Since $\mu_n = \nu_n + \omega_n \uparrow \mu$, it follows that $\mu = \nu + \omega$. The relation $\nu_n \ll \lambda$ for each n easily implies $\nu \ll \lambda$. In view of $\omega_n \perp \lambda = 0$ for each n, it follows from Lemma 37.6 that $\omega \perp \lambda = 0$. That is, $\nu \ll \lambda$ and $\omega \perp \lambda$ both hold, and so $\mu = \nu + \omega$ is the Lebesgue decomposition of μ with respect to λ.

From Problem 39.1 (or by repeating the proof of Theorem 39.6), we see that $D\mu_n(x) = D\nu_n(x)$ and $D\mu(x) = D\nu(x)$ both hold for almost all x. Let $D\mu_n = \frac{d\mu_n}{d\lambda} = f_n$ for each n. In view of $\mu_n(\mathbf{R}^k) = \int f_n\,d\lambda \le \mu(\mathbf{R}^k) < \infty$, Levi's Theorem 22.8 shows that there exists some $f \in L_1(\mathbf{R}^k)$ with $f_n \uparrow f$. Now note that $\nu_n(E) = \int_E f_n\,d\lambda$ implies $\nu(E) = \int_E f\,d\lambda$ for each Borel set E. This implies $f = D\nu$ a.e., and so

$$D\mu_n(x) = D\nu_n(x) = f_n(x) \uparrow f(x) = D\nu(x) = D\mu(x)$$

holds for almost all x, as desired.

Problem 39.7. *This problem reveals some basic properties of functions of bounded variation on an interval* $[a, b]$.

a. *If* f *is differentiable at every point and* $|f'(x)| \le M < \infty$ *holds for all* $x \in [a, b]$, *then show that* f *is absolutely continuous (and hence, of bounded variation).*

b. *Show that the function* $f\colon [0, 1] \longrightarrow \mathbf{R}$ *defined by*

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ x^2\cos(\frac{1}{x^2}) & \text{if } 0 < x \le 1 \end{cases}$$

is differentiable at each x, *but is not of bounded variation (and hence,* f *is continuous but not absolutely continuous).*

c. *If* f *is a function of bounded variation and* $|f(x)| \ge M > 0$ *holds for each* $x \in [a, b]$, *then show that* $g(x) = \frac{1}{f(x)}$ *is a function of bounded variation.*

d. *If a function* $f:[a,b] \longrightarrow \mathbb{R}$ *satisfies a Lipschitz condition (i.e., if there exists a real number* M *such that* $|f(x)-f(y)| \le M|x-y|$ *holds for all* $x, y \in [a,b]$*), then show that* f *is absolutely continuous.*

Solution. (a) If $(a_1,b_1),\ldots,(a_n,b_n)$ are pairwise disjoint open subintervals of $[a,b]$, then by the Mean Value Theorem we have

$$\sum_{i=1}^{n}|f(b_i)-f(a_i)| \le M\sum_{i=1}^{n}(b_i-a_i).$$

The preceding easily implies that f is an absolutely continuous function.
(b) Only the differentiability of f at zero needs verification. The inequality

$$\left|\tfrac{f(x)-f(0)}{x-0}\right| = \left|x\cos\left(\tfrac{1}{x^2}\right)\right| \le x$$

for $0 < x \le 1$ yields $f'(0)=0$. Now if

$$P_n = \left\{0, \sqrt{\tfrac{2}{2n\pi}}, \sqrt{\tfrac{2}{(2n-1)\pi}}, \ldots, \sqrt{\tfrac{2}{3\pi}}, \sqrt{\tfrac{2}{2\pi}}, 1\right\},$$

then an easy computation shows that the variation of f with respect to the partition P_n is

$$\cos 1 + \tfrac{2}{\pi}\cdot\sum_{k=1}^{n}\tfrac{1}{k} \le V_f.$$

This implies $V_f = \infty$.
(c) Note that for each $a \le x < y \le b$, we have

$$|g(x)-g(y)| = \tfrac{|f(x)-f(y)|}{|f(x)f(y)|} \le \tfrac{1}{M^2}|f(x)-f(y)|.$$

Therefore, $V_g \le \frac{1}{M^2}V_f < \infty$ holds.
(d) Let a function $f:[a,b] \longrightarrow \mathbb{R}$ satisfy a Lipschitz condition as stated in the problem and let $\varepsilon > 0$. Put $\delta = \frac{\varepsilon}{M} > 0$ and note that if $(a_1,b_1),\ldots,(a_n,b_n)$ are pairwise disjoint open subintervals of $[a,b]$ satisfying $\sum_{i=1}^{n}(b_i-a_i) < \delta$, then

$$\begin{aligned}\sum_{i=1}^{n}|f(b_i)-f(a_i)| &\le \sum_{i=1}^{n}M(b_i-a_i)\\ &= M\sum_{i=1}^{n}(b_i-a_i) < M\delta = \varepsilon.\end{aligned}$$

Problem 39.8. *This problem presents an example of a continuous increasing function (and hence, of bounded variation) that is not absolutely continuous.*

Consider the Cantor set C as constructed in Example 6.15 of the text. Recall that C was obtained from $[0, 1]$ by removing certain open intervals by steps. In the first step we removed the open middle third interval. At the nth step there were 2^{n-1} closed intervals, all of the same length, and we removed the open middle third interval from each one of them. Let us denote by $I_1^n, \ldots, I_{2^{n-1}}^n$ (counted from left to right) the removed open intervals at the nth step. Now, define the function $f: [0, 1] \to [0, 1]$ as follows:

i. $f(0) = 0$;
ii. *if $x \in I_i^n$ for some $1 \le i \le 2^{n-1}$, then $f(x) = (2i-1)/2^n$; and*
iii. *if $x \in C$ with $x \neq 0$, then $f(x) = \sup\{f(t): t < x \text{ and } t \in [0, 1] \setminus C\}$.*

Part of the graph of f is shown in Figure 7.1.

a. *Show that f is an increasing continuous function from $[0, 1]$ to $[0, 1]$.*
b. *Show that $f'(x) = 0$ for almost all x.*
c. *Show that f is not absolutely continuous.*
d. *Show that $\mu_f \perp \lambda$ holds.*

Solution. Notice again that parts of the graph of the function f are shown in Figure 7.1.
(a) Straightforward.
(b) Observe that f is constant on each I_i^n. This implies that $f'(x) = 0$ holds

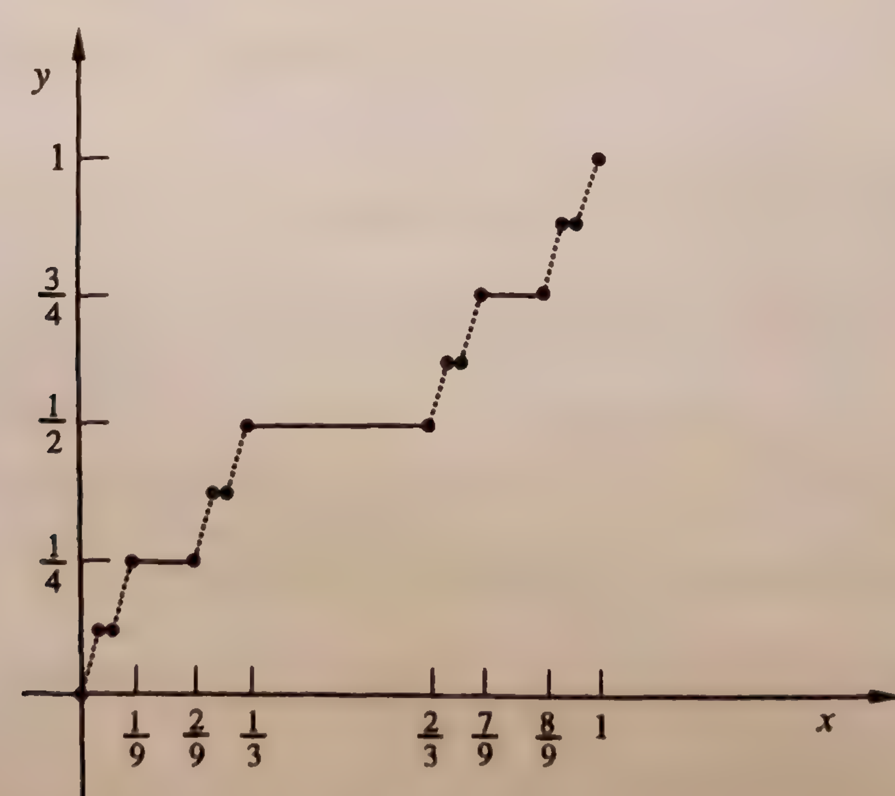

FIGURE 7.1.

for all $x \in [0,1] \setminus C$. Since $\lambda(C) = 0$, it follows that $f'(x) = 0$ holds for almost all x.
(c) If f is absolutely continuous, then by Theorem 39.15 we should have

$$1 = f(1) - f(0) = \int_0^1 f'(x)\,d\lambda(x) = 0,$$

which is impossible.
(d) Note that if $B = [0,1] \setminus C$, then $B \cup C = [0,1]$ and $\mu_f(B) = \lambda(C) = 0$. Hence, $\mu_f \perp \lambda$ holds.

Problem 39.9. *Let $f\colon [a,b] \to \mathbf{R}$ be an absolutely continuous function. Then show that f is a constant function if and only if $f'(x) = 0$ holds for almost all x.*

Solution. Assume that $f\colon [a,b] \longrightarrow \mathbf{R}$ is an absolutely continuous function such that $f'(x) = 0$ holds for almost all x. By Theorem 39.15, we have

$$f(x) - f(a) = \int_a^x f'(t)\,d\lambda(t) = 0$$

for each $x \in [a,b]$. Hence, $f(x) = f(a)$ holds for each $x \in [a,b]$, so that f is a constant function.

Problem 39.10. *Let f and g be two left continuous functions (on $\mathbf{R}$). Show that $\mu_f = \mu_g$ holds if and only if $f - g$ is a constant function.*

Solution. If $f - g$ is a constant, then it is easy to see that $\mu_f = \mu_g$ holds. For the converse, assume $\mu_f = \mu_g$. If $x > 0$, then

$$f(x) - f(0) = \mu_f\big([0,x)\big) = \mu_g\big([0,x)\big) = g(x) - g(0)$$

implies $f(x) - g(x) = f(0) - g(0)$. Similarly, if $x < 0$, then

$$f(0) - f(x) = \mu_f\big([x,0)\big) = \mu_g\big([x,0)\big) = g(0) - g(x),$$

and so $f(x) - g(x) = f(0) - g(0)$ holds in this case too.

Problem 39.11. *This problem presents another characterization of the norm dual of $C[a,b]$. Start by letting L denote the collection of all functions of bounded variation on $[a,b]$ that are left continuous and vanish at a.*

a. *Show that L under the usual algebraic operations is a vector space, and that $f \mapsto \mu_f$, from L to $M_b([a,b])$, is linear, one-to-one, and onto.*

b. *Define $f \succeq g$ to mean that $f-g$ is an increasing function. (Note that $f \geq g$ does not imply $f \succeq g$.) Show that L under $\succeq$ is a partially ordered vector space such that $f \succeq g$ holds in L if and only if $\mu_f \geq \mu_g$ in $M_b([a,b])$.*

c. *Establish that L with the norm $\|f\| = V_{|f|}$ is a Banach lattice.*

d. *Show, with an appropriate interpretation, that $C^*[a,b] = L$.*

Solution. (a) Clearly, L under the usual algebraic operations is a vector space. Also, it should be clear that $f \longmapsto \mu_f$ from L to $M_b([a,b])$ is a linear mapping.

To see that $f \longmapsto \mu_f$ is one-to-one, assume $\mu_f = 0$. Then,

$$f(x) = f(x) - f(a) = \mu_f([a,x)) = 0$$

holds for all $a < x \leq b$, and so $f = 0$.

Next, we shall show that $\mu \longmapsto \mu_f$ is onto. Assume at the beginning that $0 \leq \mu \in M_b([a,b])$. Define the function

$$f(x) = \begin{cases} 0 & \text{if } x \leq a \\ \mu([a,x)) & \text{if } a < x < b \\ \mu([a,b]) & \text{if } x \geq b\,, \end{cases}$$

and note that f is increasing, left continuous, and satisfies $f(a) = 0$. Thus, $f \in L$. Now, an easy argument shows that $\mu = \mu_f$. Finally, if $\mu \in M_b([a,b])$, then pick two increasing functions $f, g \in L$ with $\mu^+ = \mu_f$ and $\mu^- = \mu_g$. Note that the function $h = f - g \in L$ satisfies $\mu = \mu^+ - \mu^- = \mu_f - \mu_g = \mu_{f-g} = \mu_h$.

(b) Straightforward.

(c) Since $f \succeq g$ holds in L if and only if $\mu_f \geq \mu_g$ holds in $M_b([a,b])$ and $M_b([a,b])$ is a vector lattice, it is easy to see that L must likewise be a vector lattice. Moreover, by Lemma 38.6 the mapping $f \longmapsto \mu_f$ is a lattice isomorphism from L onto $M_b([a,b])$.

Now, note that if $f \succeq 0$ holds in L (i.e., if f is an increasing function), then

$$V_f = f(b) - f(a) = \mu_f([a,b]) = \|\mu_f\|$$

holds. Thus, for each $f \in L$ we have

$$\|\mu_f\| = \||\mu_f|\| = \|\mu_{|f|}\| = V_{|f|}.$$

This implies that $\|f\| = V_{|f|}$ defines a lattice norm on L, and that $f \longmapsto \mu_f$ from L onto $M_b([a,b])$ is a lattice isometry. In particular, L with the norm $\|f\| = V_{|f|}$ is a Banach lattice.

(d) Using the notation of Theorem 38.7, we see that the composition of the two operators

$$f \longmapsto \mu_f \longmapsto F_{\mu_f}$$

is a lattice isometry from L onto $C^*[a,b]$.

Problem 39.12. *If $f:[a,b] \to \mathbb{R}$ is an increasing function, then show that f' is Lebesgue integrable and that $\int_a^b f'(x)\,dx \le f(b) - f(a)$ holds. Give an example of an increasing function f for which $\int_a^b f'(x)\,dx < f(b) - f(a)$ holds.*

Solution. Let $g_n(x) = n[f(x+\frac{1}{n}) - f(x)]$ for each $x \in [a,b]$ (where, of course, $f(x) = f(b)$ for $x > b$.) Clearly, $g_n(x) \longrightarrow f'(x)$ holds for almost all x; see Theorem 39.9. On the other hand, the relation $g_n(x) \ge 0$ for each x, and

$$\begin{aligned}
\int_a^b g_n(x)\,dx &= n\Big[\int_a^b f\big(x+\tfrac{1}{n}\big)\,dx - \int_a^b f(x)\,dx\Big]\\
&= n\Big[\int_{a+\frac{1}{n}}^{b+\frac{1}{n}} f(x)\,dx - \int_a^b f(x)\,dx\Big]\\
&= n\Big[\int_b^{b+\frac{1}{n}} f(x)\,dx - \int_a^{a+\frac{1}{n}} f(x)\,dx\Big]\\
&= n\Big[\tfrac{1}{n}f(b) - \int_a^{a+\frac{1}{n}} f(x)\,dx\Big]\\
&\le n\big[\tfrac{1}{n}f(b) - \tfrac{1}{n}f(a)\big] = f(b) - f(a),
\end{aligned}$$

coupled with Fatou's Lemma show that $f' \in L_1\big([a,b]\big)$ and that

$$\int_a^b f'(x)\,dx = \int_a^b \lim_{n\to\infty} g_n(x)\,dx \le \liminf_{n\to\infty} \int_a^b g_n(x)\,dx \le f(b) - f(a).$$

Finally, an example of a function that yields strict inequality is provided by the function described in Problem 39.8.

Problem 39.13. *If $f:[a,b] \to \mathbb{R}$ is an absolutely continuous function, then show that*

$$V_f = \int_a^b |f'(x)|\,dx$$

holds.

Solution. By Theorem 39.15 we have $f' \in L_1\big([a,b]\big)$ and

$$\mu_f(E) = \int_E f'(x)\,dx$$

holds for each Borel subset E of $[a,b]$.

Start by observing that if $a = t_0 < t_1 < \cdots < t_n = b$ is a partition of $[a, b]$, then

$$\sum_{i=1}^{n} |f(t_i) - f(t_{i-1})| = \sum_{i=1}^{n} \Big| \int_{t_{i-1}}^{t_i} f'(x)\,dx \Big|$$
$$\le \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} |f'(x)|\,dx = \int_a^b |f'(x)|\,dx.$$

Therefore, $V_f \le \int_a^b |f'(x)|\,dx$ holds. Now, since the continuous functions are dense (in the L_1-norm) in $L_1([a, b])$ (Theorem 25.3) and the functions of the form

$$\phi = \sum_{i=1}^{n} a_i \chi_{(t_{i-1}, t_i)},$$

where $a = t_0 < t_1 < \cdots < t_n = b$ is a partition of $[a, b]$, are dense (in the L_1-norm) in $C[a, b]$, these functions are also dense in $L_1([a, b])$. Thus, given $\varepsilon > 0$, there exist a partition $a = t_0 < t_1 < \cdots < t_n = b$ and real numbers $a_1, \ldots, a_n$ so that $\phi = \sum_{i=1}^{n} a_i \chi_{(t_{i-1}, t_i)}$ satisfies $\|\phi - \operatorname{Sgn} f'\|_1 < \varepsilon$. In view of $\big|(-1 \vee \phi) \wedge 1 - \operatorname{Sgn} f'\big| \le |\phi - \operatorname{Sgn} f'|$, we can assume that $|\phi(x)| \le 1$ holds for all $x \in [a, b]$. Moreover, we have

$$\int_a^b \phi(x) f'(x)\,dx = \sum_{i=1}^{n} a_i \int_{t_{i-1}}^{t_i} f'(x)\,dx = \sum_{i=1}^{n} a_i \big[f(t_i) - f(t_{i-1})\big]$$
$$\le \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})| \le V_f.$$

Next, choose a sequence $\{\phi_n\}$ of step functions of the previous type satisfying $\phi_n \longrightarrow \operatorname{Sgn} f'$ a.e. (see Lemma 31.6 of the text). In view of $|\phi_n f'| \le |f'|$, the Lebesgue Dominated Convergence Theorem implies

$$\int_a^b |f'(x)|\,dx = \int_a^b f'(x) \cdot \operatorname{Sgn} f'(x)\,dx$$
$$= \lim_{n\to\infty} \int_a^b \phi_n(x) f'(x)\,dx \le V_f.$$

Thus, $V_f = \int_a^b |f'(x)|\,dx$ holds.

It is interesting to observe that $V_f = |\mu_f|([a,b])$ also holds. To see this, let $a = t_0 < t_1 < \cdots < t_n = b$ be an arbitrary partition of $[a,b]$. Then, we have

$$\sum_{i=1}^{n}|f(t_i) - f(t_{i-1})| = \sum_{i=1}^{n}|\mu_f([t_{i-1}, t_i))| \\ \leq \sum_{i=1}^{n}|\mu_f|([t_{i-1}, t_i)) = |\mu_f|([a,b]),$$

and so $V_f \leq |\mu_f|([a,b])$. On the other hand, if $E_1, \ldots, E_n$ are pairwise disjoint Borel subsets of $[a,b]$, then

$$\sum_{i=1}^{n}|\mu_f(E_i)| = \sum_{i=1}^{n}\left|\int_{E_i} f'(x)\,dx\right| \leq \sum_{i=1}^{n}\int_{E_i}|f'(x)|\,dx \\ \leq \int_a^b |f'(x)|\,dx = V_f$$

holds, which (by Theorem 36.9) implies that $|\mu_f|([a,b]) \leq V_f$. Consequently, $|\mu_f|([a,b]) = V_f$ holds.

Problem 39.14. *For a continuously differentiable function $f\colon [a,b] \to \mathbb{R}$ establish the following properties:*

a. *The signed measure μ_f is absolutely continuous with respect to the Lebesgue measure and $d\mu_f/d\lambda = f'$ a.e.*
b. *If $g\colon [a,b] \to \mathbb{R}$ is Riemann integrable, then gf' is also Riemann integrable and*

$$\int g\,d\mu_f = \int_a^b g(x)f'(x)\,dx.$$

Solution. (a) By Problem 39.7, we know that f is absolutely continuous and so (by Theorem 39.12) μ_f is absolutely continuous with respect to the Lebesgue measure. Now, combining Theorems 39.14(2), 39.8, and 39.4, we see that

$$\tfrac{d\mu_f}{d\lambda} = D\mu_f = f' \text{ a.e.}$$

(b) Since f' is a continuous function and g is Riemann integrable, it follows that gf' is also Riemann (and hence Lebesgue) integrable over $[a,b]$. From

$\mu_f(A) = \int_A f'\,d\lambda$ for every Borel subset A of $[a, b]$ and Problem 22.15, we see that

$$\int_{[a,b]} g\,d\mu_f = \int_{[a,b]} gf'\,d\lambda = \int_a^b g(x)f'(x)\,dx.$$

Problem 39.15. *For each n consider the increasing continuous function $f_n\colon \mathbf{R} \to \mathbf{R}$ defined by*

$$f_n(x) = \begin{cases} 1 & \text{if } x > 0, \\ n(x-1)+1 & \text{if } 1-\frac{1}{n} < x < 1, \\ 0 & \text{if } x \le 1-\frac{1}{n}. \end{cases}$$

If $f\colon \mathbf{R} \to \mathbf{R}$ is a continuous function, then show that

a. *f is μ_{f_n}-integrable for each n, and*
b. $\lim \int f\,d\mu_{f_n} = f(1)$.

Solution. Note that $\operatorname{Supp}\mu_{f_n} = [1-\frac{1}{n}, 1]$. This easily implies that f is μ_{f_n}-integrable for each n. In addition, note that $f_n'(x) = n$ holds for each $1-\frac{1}{n} \le x \le 1$. By the preceding problem, we see that

$$\int f\,d\mu_{f_n} = \int_{1-\frac{1}{n}}^1 f(x)f_n'(x)\,dx = \int_{1-\frac{1}{n}}^1 nf(x)\,dx = \frac{\int_{1-\frac{1}{n}}^1 f(x)\,dx}{\frac{1}{n}}.$$

Therefore, by the Fundamental Theorem of Calculus, we infer that $\lim \int f\,d\mu_{f_n} = f(1)$.

Problem 39.16. *Let $f\colon \mathbf{R} \to \mathbf{R}$ be a (uniformly) bounded function and let*

$$E = \{x \in \mathbf{R}\colon\ f'(x) \text{ exists in } \mathbf{R}\}.$$

If $\lambda(E) = 0$, then show that $\lambda(f(E)) = 0$.

Solution. For each natural number n, let

$$E_n = \{a \in E\colon\ |f(x) - f(a)| \le n|x-a| \text{ for all } x \in \mathbf{R}\}.$$

Since f is bounded, it is easy to see that $E = \bigcup_{n=1}^\infty E_n$, and so $f(E) = \bigcup_{n=1}^\infty f(E_n)$ (see Problem 1.1(6)). Thus, in order to establish that $\lambda(f(E)) = 0$, it suffices to show that $\lambda(f(E_n)) = 0$ holds for each n. To this end, fix n and $\varepsilon > 0$.

From $\lambda(E) = 0$, we obtain $\lambda(E_n) = 0$, and so there exists a sequence of open intervals $\{(b_k - r_k, b_k + r_k)\}$ satisfying

$$E_n \subseteq \bigcup_{k=1}^{\infty}(b_k - r_k, b_k + r_k) \quad \text{and} \quad 2\sum_{k=1}^{\infty} r_k < \varepsilon.$$

Now, note that if $a \in E_n$, then there exists some m with $|b_m - a| < r_m$, and hence $|f(b_m) - f(a)| \leq n|b_m - a| < nr_m$ holds. It follows that $f(E_n) \subseteq \bigcup_{k=1}^{\infty}\big(f(b_k) - nr_k, f(b_k) + nr_k\big)$. Therefore,

$$\sum_{k=1}^{\infty} \lambda\big((f(b_k) - nr_k, f(b_k) + nr_k)\big) = 2n\sum_{k=1}^{\infty} r_k < 2n\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we infer that $\lambda\big(f(E_n)\big) = 0$, as desired. (Compare this problem with Problem 18.9.)

Problem 39.17. *This problem presents an example of a continuous function $f: \mathbb{R} \to \mathbb{R}$ which is nowhere differentiable; this example should be compared with Problem 9.28. Consider the function ϕ: $[0, 2] \to \mathbb{R}$ defined by $\phi(x) = x$ if $0 \leq x \leq 1$ and $\phi(x) = 2 - x$ if $1 < x \leq 2$. Extend ϕ to all of $\mathbb{R}$ (periodically) so that $\phi(x) = \phi(x + 2)$ holds for all $x \in \mathbb{R}$. Now, define the function f: $\mathbb{R} \to \mathbb{R}$ by*

$$f(x) = \sum_{n=0}^{\infty} \left(\tfrac{3}{4}\right)^n \phi(4^n x).$$

Show that f is a continuous nowhere differentiable function.

Solution. Since the series $\sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n$ converges and $0 \leq \phi(x) \leq 1$ holds for all x, it is easy to see that the sequence of partial sums of the series $f(x) = \sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n \phi(4^n x)$ converges uniformly to f on $\mathbb{R}$. So, by Theorem 9.2, f is a well-defined continuous function.

Now, fix $x_0 \in \mathbb{R}$. The proof of the nondifferentiability of f at x_0 will be based upon the following property of differentiable functions.

- *If $h: (a, b) \to \mathbb{R}$ is differentiable at some $x_0 \in (a, b)$ and $\mu = h'(x_0)$, then for each $\epsilon > 0$ there exists some $\delta > 0$ such that whenever $x, y \in (a, b)$ satisfy $x < x_0 < y$ and $y - x < \delta$, then $\left|\frac{h(y) - h(x)}{y - x} - \mu\right| < \epsilon$.*

This conclusion follows easily from the inequalities

$$\begin{aligned}\left|\frac{h(y)-h(x)}{y-x}-\mu\right| &= \left|\frac{[h(y)-h(x_0)-\mu(y-x_0)]+[h(x_0)-h(x)-\mu(x_0-x)]}{y-x}\right|\\ &\le \left|\frac{h(y)-h(x_0)-\mu(y-x_0)}{y-x_0}\right|\cdot\left|\frac{y-x_0}{y-x}\right|+\left|\frac{h(x_0)-h(x)-\mu(x_0-x))}{x_0-x}\right|\cdot\left|\frac{x_0-x}{y-x}\right|\\ &\le \left|\frac{h(y)-h(x_0)}{y-x_0}-\mu\right|+\left|\frac{h(x_0)-h(x)}{x_0-x}-\mu\right|.\end{aligned}$$

Now, for each natural number m, then there exists a unique integer k_m such that $k_m \le 4^m x_0 < k_m + 1$. Let

$$s_m = 4^{-m}k_m \quad \text{and} \quad t_m = 4^{-m}(k_m+1),$$

and note that $s_m \le x_0 < t_m$ holds for each m. From $t_m - s_m = 4^{-m}$, we see that $\lim t_m = \lim s_m = x_0$.

The reader should keep in mind that if p and q are two integers, then $\phi(p) - \phi(q) = 0$ if $p-q$ is an even integer and $|\phi(p)-\phi(q)| = 1$ if $p-q$ is an odd integer. Next, observe that if n is a non-negative integer, then $4^n t_m - 4^n s_m = 4^{n-m}$. So, from the definition of ϕ, we have:

a. if $n > m$, then $\phi(4^n t_m) - \phi(4^n s_m) = 0$,
b. if $n = m$, then $\phi(4^n t_m) - \phi(4^n s_m) = 1$, and
c. if $0 \le n < m$, then $\phi(4^n t_m) - \phi(4^n s_m) = 4^{n-m}$.

Therefore, for each m we have

$$\begin{aligned}|f(t_m)-f(s_m)| &= \left|\sum_{n=0}^{\infty}\left(\tfrac{3}{4}\right)^n[\phi(4^n t_m)-\phi(4^n s_m)]\right|\\ &= \left|\sum_{n=0}^{m}\left(\tfrac{3}{4}\right)^n[\phi(4^n t_m)-\phi(4^n s_m)]\right|\\ &\ge \left(\tfrac{3}{4}\right)^m - \sum_{n=0}^{m-1}\left(\tfrac{3}{4}\right)^n 4^{n-m} > \tfrac{1}{2}\left(\tfrac{3}{4}\right)^m.\end{aligned}$$

This implies $\left|\frac{f(t_m)-f(s_m)}{t_m-s_m}\right| > \frac{3^m}{2}$ for each m. Now, a glance at (•) shows that f cannot be differentiable at x_0. Since x_0 is arbitrary, f is differentiable at no point of $\mathbb{R}$.

40. THE CHANGE OF VARIABLES FORMULA

Problem 40.1. *Show that an open ball in a Banach space is a connected set. That is, show that if B is an open ball in a Banach space such that $B = \mathcal{O}_1 \cup \mathcal{O}_2$ holds with both $\mathcal{O}_1$ and $\mathcal{O}_2$ open and disjoint, then either $\mathcal{O}_1 = \emptyset$ or $\mathcal{O}_2 = \emptyset$.*

Solution. Let B be an open ball in a Banach space. Assume by way of contradiction that there exist two nonempty open sets O_1 and O_2 such that $B = O_1 \cup O_2$ and $O_1 \cap O_2 = \emptyset$. Fix two elements $a \in O_1$ and $b \in O_2$, and then define the function $f\colon [0,1] \longrightarrow B$ by $f(t) = ta + (1-t)b$. Clearly, $f(0) = b$ and $f(1) = a$. Moreover, in view of the inequality

$$\|f(t) - f(s)\| = \|(t-s)a + (s-t)b\| \le \big(\|a\| + \|b\|\big)|t-s|,$$

we see that f is a (uniformly) continuous function.

Let $\alpha = \inf\{t \in [0,1]\colon f(t) \in O_1\}$. Choose a sequence $\{\alpha_n\}$ of $[0,1]$ with $\alpha_n \to \alpha$ and $f(\alpha_n) \in O_1$ for each n. By the continuity of f we have $f(\alpha_n) \to f(\alpha)$. Since O_2 is open and disjoint from O_1, it follows that $f(\alpha) \notin O_2$. In particular, $\alpha > 0$ must hold. Thus, there exists a sequence $\{\beta_n\}$ of real numbers with $0 < \beta_n < \alpha$ for each n and $\beta_n \longrightarrow \alpha$. By the definition of α, we see that $f(\beta_n) \in O_2$ holds for each n, and hence, as above $f(\alpha) \notin O_1$. Now, note that

$$f(\alpha) \notin O_1 \cup O_2 = B$$

holds, which is impossible.

Problem 40.2. *Let $T\colon V \to \mathbf{R}^k$ be C^1-differentiable. Show that the mapping $x \mapsto T'(x)$ from V into $L(\mathbf{R}^k, \mathbf{R}^k)$ is a continuous function.*

Solution. We know that

$$T'(x) = \begin{bmatrix} \frac{\partial T_1}{\partial x_1}(x) & \cdots & \frac{\partial T_1}{\partial x_k}(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial T_k}{\partial x_1}(x) & \cdots & \frac{\partial T_k}{\partial x_k}(x) \end{bmatrix}.$$

So, if $a = (a_1, \ldots, a_k) \in \mathbf{R}^k$ satisfies $\|a\|_2 = \big(\sum_{i=1}^k a_i^2\big)^{\frac{1}{2}} = 1$, then using the

Cauchy–Schwarz inequality, we see that

$$\begin{aligned}\|[T'(x)-T'(y)]a\|_2 &= \Big(\sum_{i=1}^{k}\Big[\sum_{j=1}^{k}\big(\tfrac{\partial T_i}{\partial x_j}(x)-\tfrac{\partial T_i}{\partial x_j}(y)\big)\cdot a_j\Big]^2\Big)^{\frac{1}{2}}\\ &\le \Big[\sum_{i=1}^{k}\Big(\sum_{j=1}^{k}\big[\tfrac{\partial T_i}{\partial x_j}(x)-\tfrac{\partial T_i}{\partial x_j}(y)\big]^2\Big)\cdot\Big(\sum_{j=1}^{k}a_j^2\Big)\Big]^{\frac{1}{2}}\\ &= \Big(\sum_{i=1}^{k}\sum_{j=1}^{k}\big[\tfrac{\partial T_i}{\partial x_j}(x)-\tfrac{\partial T_i}{\partial x_j}(y)\big]^2\Big)^{\frac{1}{2}}.\end{aligned}$$

Consequently,

$$\begin{aligned}\|T'(x)-T'(y)\| &= \sup\big\{\|[T'(x)-T'(y)]a\|_2\colon \|a\|_2=1\big\}\\ &\le \Big(\sum_{i=1}^{k}\sum_{j=1}^{k}\big[\tfrac{\partial T_i}{\partial x_j}(x)-\tfrac{\partial T_i}{\partial x_j}(y)\big]^2\Big)^{\frac{1}{2}}\end{aligned}$$

for each pair $x,\ y\in V$. This inequality, coupled with the fact that $T\colon V\longrightarrow \mathbf{R}^k$ is C^1-differentiable, implies that $x\longmapsto T'(x)$ from V into $L(\mathbf{R}^k,\mathbf{R}^k)$ is a continuous function.

Problem 40.3. *Show that the Lebesgue measure on $\mathbf{R}^2$ is "rotation" invariant.*

Solution. A "*rotation*" of the plane is a linear operator $T\colon \mathbf{R}^2\longrightarrow\mathbf{R}^2$ whose representing matrix A is orthogonal (i.e., it satisfies $AA'=A'A=I$). Any such orthogonal matrix is of the form

$$A=\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix},$$

where θ represents the angle of rotation; see Figure 7.2.

In particular, note that $\det A=1$. Thus, by Lemma 40.4, we see that

$$\lambda\big(A(E)\big)=\big|\det A\big|\lambda(E)=\lambda(E)$$

holds for each Lebesgue measurable subset E of $\mathbf{R}^2$.

Problem 40.4 (Polar Coordinates). *Let*

$$E=\{(r,\theta)\in\mathbf{R}^2\colon\ r\ge 0 \text{ and } 0\le\theta\le 2\pi\}.$$

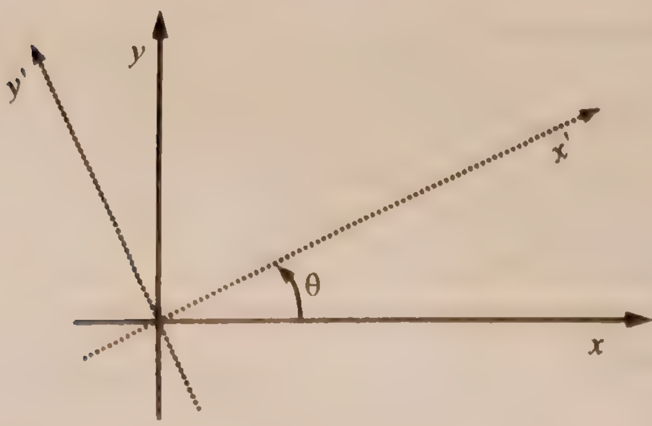

FIGURE 7.2. Rotation by an Angle θ

The transformation $T: E \to \mathbb{R}^2$ *defined by* $T(r, \theta) = (r\cos\theta, r\sin\theta)$, *or as it is usually written*

$$x = r\cos\theta \quad \textit{and} \quad y = r\sin\theta,$$

is called the **polar coordinate transformation** *on* $\mathbb{R}^2$, *shown graphically in Figure 7.3.*

a. *Show that* $\lambda(E \setminus E^o) = 0$.
b. *If* $A = \{(x, 0)\colon\ x \geq 0\}$, *then show that* A *is a closed subset of* $\mathbb{R}^2$ *whose (2-dimensional) Lebesgue measure is zero.*
c. *Show that* $T: E^o \to \mathbb{R}^2 \setminus A$ *is a diffeomorphism whose Jacobian determinant satisfies* $J_T(r, \theta) = r$ *for each* $(r, \theta) \in E^o$.
d. *Show that if* G *is a Lebesgue measurable subset of* E *with* $\lambda(G \setminus G^o) = 0$, *then* $T(G)$ *is a Lebesgue measurable subset of* $\mathbb{R}^2$. *Moreover, show that if* $f \in L_1\big(T(G)\big)$, *then*

$$\int_{T(G)} f\,d\lambda = \int\!\!\int_G f(r\cos\theta, r\sin\theta) r\,dr\,d\theta$$

holds.

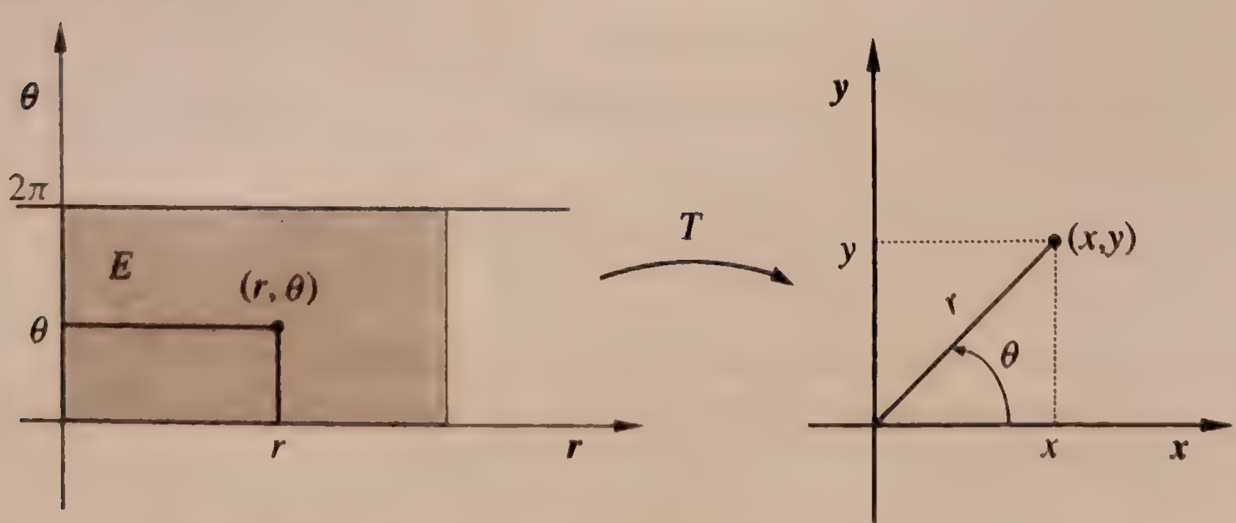

FIGURE 7.3. The Polar Coordinate Transformation

Solution. (a) If we consider the sets $X = \{(r, 0)\colon r \geq 0\}$, $Y = \{(r, 2\pi)\colon r \geq 0\}$, and $Z = \{(0, \theta)\colon 0 \leq \theta \leq 2\pi\}$, then $E \setminus E^o = X \cup Y \cup Z$. To show that $\lambda(E \setminus E^o) = 0$, it suffices to establish that $\lambda(X) = \lambda(Y) = \lambda(Z) = 0$.

Let $X_n = \{(r, 0)\colon 0 \leq r \leq n\}$ and $Y_n = \{(r, 2\pi)\colon 0 \leq r \leq n\}$. In view of $X_n \subseteq [0, n] \times [-\varepsilon, \varepsilon]$ and $Y_n \subseteq [0, n] \times [2\pi - \varepsilon, 2\pi + \varepsilon]$, we see that $\lambda(X_n) = \lambda(Y_n) = 0$ holds for each n. Since $X_n \uparrow X$ and $Y_n \uparrow Y$, it follows that $\lambda(X) = \lambda(Y) = 0$.

Also, the inclusion $Z \subseteq [-\varepsilon, \varepsilon] \times [0, 2\pi]$ implies $\lambda(Z) \leq 4\pi\varepsilon$ for each $\varepsilon > 0$, and thus $\lambda(Z) = 0$.

(b) This is proven in part (a) previously.

(c) Clearly, $T\colon E^o \longrightarrow \mathbb{R}^2 \setminus A$ is one-to-one, onto, and C^1-differentiable. The Jacobian determinant is

$$J_T(r, \theta) = \det\begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{bmatrix} = \det\begin{bmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{bmatrix} = r,$$

which implies that $J_T(r, \theta) = r \neq 0$ holds for each $(r, \theta) \in E^o$. The preceding are enough to guarantee that $T\colon E^o \longrightarrow \mathbb{R}^2 \setminus A$ is a diffeomorphism.

(d) Clearly, $G^o \subseteq E^o$. Thus, by part (c), $T(G^o)$ is an open subset of $\mathbb{R}^2 \setminus A$ and $T\colon G^o \longrightarrow T(G^o)$ is a diffeomorphism.

Since $T\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ (defined by $T(r, \theta) = (r\cos\theta, r\sin\theta)$) is a C^1-diffeomorphism, it follows from Lemma 40.1 that $\lambda\big(T(G \setminus G^o)\big) = 0$. Now, if we consider the sets $A = G$, $B = T(G)$, $V = G^o$, and $W = T(G^o)$, then $\lambda(A \setminus V) = \lambda(G \setminus G^o) = 0$ and $\lambda\big(T(G) \setminus T(G^o)\big) \leq \lambda\big(T(G \setminus G^o)\big) = 0$ both hold. Thus, Theorem 40.8 applies and gives us the desired formula.

Problem 40.5. *This problem uses polar coordinates (introduced in the preceding problem) to present an alternate proof of Euler's formula $\int_0^\infty e^{-x^2}\,dx = \sqrt{\pi}/2$.*

a. *For each $r > 0$, let $C_r = \{(x, y) \in \mathbb{R}^2\colon x^2 + y^2 \leq r^2,\ x \geq 0,\ y \geq 0\}$ and $S_r = [0, r] \times [0, r]$. Show that $C_r \subseteq S_r \subseteq C_{r\sqrt{2}}$.*

b. *If $f(x, y) = e^{-(x^2+y^2)}$, then show that*

$$\int_{C_r} f\,d\lambda \leq \int_{S_r} f\,d\lambda \leq \int_{C_{r\sqrt{2}}} f\,d\lambda,$$

where λ is the two-dimensional Lebesgue measure.

c. *Use the change of variables to polar coordinates and Fubini's Theorem to show that*

$$\int_{C_r} f_r\,d\lambda = \int_0^{\frac{\pi}{2}}\int_0^r e^{-t^2} t\,dt\,d\theta = \frac{\pi}{4}(1 - e^{-r^2}).$$

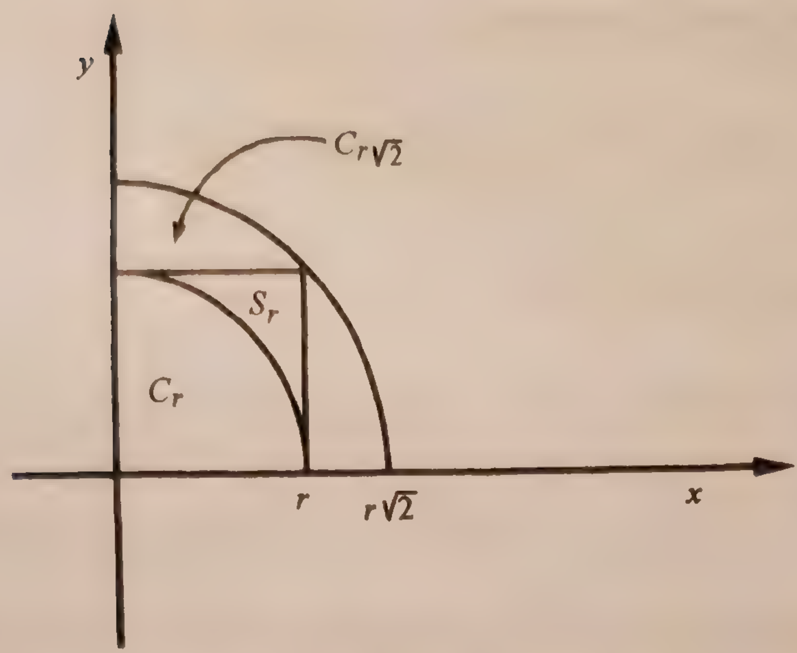

FIGURE 7.4.

d. *Use (b) to establish that*

$$\frac{\pi}{4}(1 - e^{-r^2}) \leq \left(\int_0^r e^{-x^2}\,dx\right)^2 \leq \frac{\pi}{4}(1 - e^{-2r^2}),$$

and then let $r \to \infty$ to obtain the desired formula.

Solution. (a) Geometrically the three sets are as shown in Figure 7.4.
(b) Since $f(x, y) = e^{-(x^2+y^2)} \geq 0$ holds for all (x, y), we see that

$$f\chi_{C_r} \leq f\chi_{S_r} \leq f\chi_{C_{r\sqrt{2}}},$$

and the desired inequality follows.
(c) Consider the polar coordinates transformation described in the preceding problem. For the set $G = \{(t, \theta)\colon 0 \leq t \leq r \text{ and } 0 \leq \theta \leq \frac{\pi}{2}\}$, we have

$$\begin{aligned}\int_{C_r} f\,d\lambda &= \int_{T(G)} f\,d\lambda = \int\!\!\int_G f(t\cos\theta, t\sin\theta)t\,dt\,d\theta \\ &= \int_0^{\frac{\pi}{2}}\int_0^r e^{-t^2}t\,dt\,d\theta = \tfrac{\pi}{4}\left(1 - e^{-r^2}\right).\end{aligned}$$

(d) Note that

$$\begin{aligned}\int_{C_r} f\,d\lambda = \int_0^r\int_0^r e^{-(x^2+y^2)}\,dx\,dy &= \left(\int_0^r e^{-x^2}\,dx\right)\cdot\left(\int_0^r e^{-y^2}\,dy\right) \\ &= \left(\int_0^r e^{-x^2}\,dx\right)^2.\end{aligned}$$

Thus, using (b) and (c), we see that

$$\tfrac{\pi}{4}\left(1-e^{-r^2}\right) \le \left(\int_0^r e^{-x^2}\,dx\right)^2 \le \tfrac{\pi}{4}\left(1-e^{-2r^2}\right),$$

and by letting $r \longrightarrow \infty$ we get $\left(\int_0^\infty e^{-x^2}\,dx\right)^2 = \frac{\pi}{4}$.

Problem 40.6. *In $\mathbf{R}^4$, "double" polar coordinates are defined by*

$$x = r\cos\theta,\quad y = r\sin\theta,\quad z = \rho\cos\phi,\quad w = \rho\sin\phi.$$

State the change of variables formula for this transformation, and use it to show that the "volume" of the open ball in $\mathbf{R}^4$ with center at zero and radius a is $\frac{1}{2}\pi^2 a^4$.

Solution. The transformation $T\colon \mathbf{R}^4 \longrightarrow \mathbf{R}^4$ is given by

$$T(r,\rho,\theta,\phi) = (r\cos\theta, r\sin\theta, \rho\cos\phi, \rho\sin\phi)$$

for each $(r,\rho,\theta,\phi) \in \mathbf{R}^4$. Its Jacobian determinant is

$$J_T(r,\rho,\theta,\phi) = \det\begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ 0 & 0 & \cos\phi & \sin\phi \\ -r\sin\theta & r\cos\theta & 0 & 0 \\ 0 & 0 & -\rho\sin\phi & \rho\cos\phi \end{bmatrix} = -r\rho.$$

Write $\mathbf{R}^4 = \mathbf{R}^2 \times \mathbf{R}^2$, and consider the Lebesgue measure on $\mathbf{R}^4$ as the product measure of the corresponding Lebesgue measures on the two factors. Fix $a > 0$, and let

$$E = \left\{(r,\rho)\colon r \ge 0,\ \rho \ge 0, \text{ and } r^2+\rho^2 < a^2\right\}$$

and

$$F = [0, 2\pi] \times [0, 2\pi] \subseteq \mathbf{R}^2.$$

Put $G = E \times F \subseteq \mathbf{R}^2 \times \mathbf{R}^2 = \mathbf{R}^4$, and note that $T(G) = B$, the open ball of $\mathbf{R}^4$ with center at zero and radius a. Now, if $C = \{(r,\rho)\colon r\rho = 0\} \subseteq \mathbf{R}^2$ and $D = \{(r,\rho,0,0)\colon r \ge 0,\ \rho \ge 0\} \subseteq \mathbf{R}^4$, then both sets are closed in their

corresponding spaces and their corresponding Lebesgue measures are zero. Thus, if

$$V = (E \setminus C) \times \big[(0, 2\pi) \times (0, 2\pi)\big] \text{ and } W = B \setminus D,$$

then both V and W are open subsets of $\mathbf{R}^4$ and $T: V \longrightarrow W$ is a diffeomorphism (onto). Since $\lambda(G \setminus V) = \lambda(B \setminus W) = 0$, Theorem 40.8 combined with Fubini's Theorem shows that

$$\begin{aligned}\text{Volume of } B &= \lambda(B) = \int_B d\lambda = \int_{T(G)} d\lambda = \iiiint_G r\rho\, dr\, d\rho\, d\theta\, d\phi \\ &= \int_E \Big(\int_F r\rho\, d\lambda\Big)\, d\lambda = 4\pi^2 \int_E r\rho\, dr\, d\rho \\ &= 4\pi^2 \cdot \tfrac{a^4}{8} = \tfrac{1}{2}\pi^2 a^4.\end{aligned}$$

Problem 40.7 (Cylindrical Coordinates). *Let*

$$E = \{(r, \theta, z) \in \mathbf{R}^3\colon\ r \geq 0,\ 0 \leq \theta \leq 2\pi,\ z \in \mathbf{R}\}.$$

The transformation $T: E \to \mathbf{R}^3$ defined by $T(r, \theta, z) = (r\cos\theta, r\sin\theta, z)$ or as it is usually written

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z,$$

is called the **cylindrical coordinate transformation**, *shown graphically in Figure 7.5.*

a. *Show that $\lambda(E \setminus E^o) = 0$.*
b. *If $A = \{(x, 0, z) \in \mathbf{R}^3\colon\ x \geq 0,\ z \in \mathbf{R}\}$, then show that A is a closed subset of $\mathbf{R}^3$ whose (three-dimensional) Lebesgue measure is zero.*

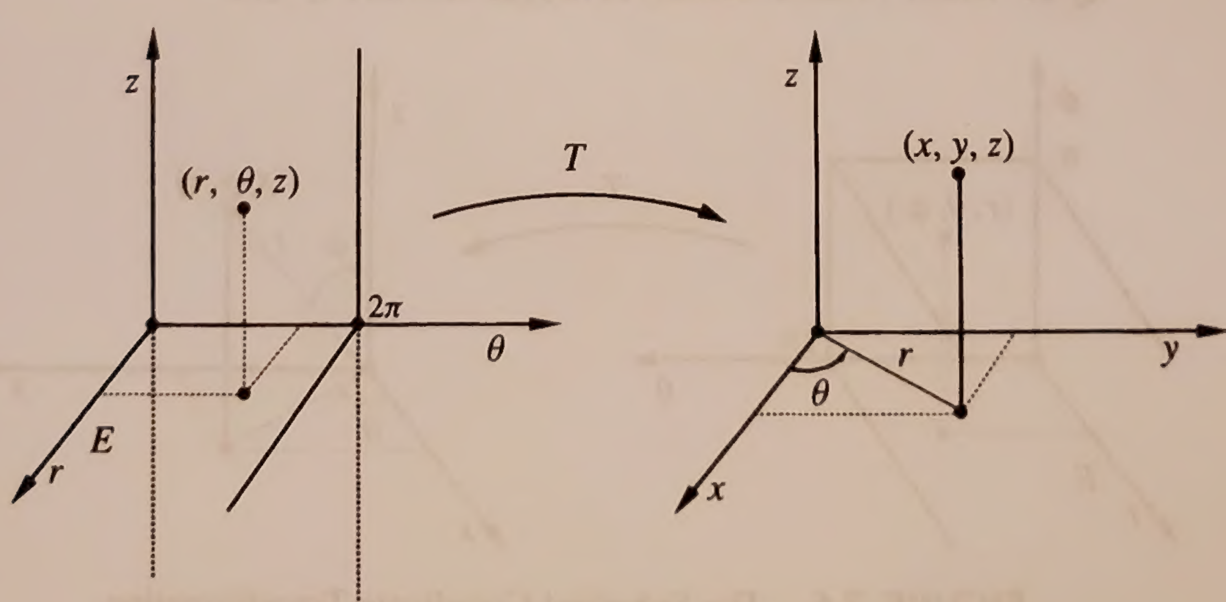

FIGURE 7.5. The Cylindrical Coordinate Transformation

c. *Show that* $T: E^o \to \mathbf{R}^3 \setminus A$ *is a diffeomorphism whose Jacobian determinant satisfies* $J_T(r, \theta, z) = r$ *for each* $(r, \theta, z) \in E^o$.

d. *Show that if* G *is a Lebesgue measurable subset of* E *with* $\lambda(G \setminus G^o) = 0$, *then* $T(G)$ *is a Lebesgue measurable subset of* $\mathbf{R}^3$. *Moreover, show that if* $f \in L_1(T(G))$, *then*

$$\int_{T(G)} f\, d\lambda = \iiint_G f(r\cos\theta, r\sin\theta, z)\, r\, dr\, d\theta\, dz$$

holds.

Solution. Repeat the solution of Problem 40.4.

Problem 40.8 (Spherical Coordinates). *Let*

$$E = \{(r, \theta, \phi) \in \mathbf{R}^3 \colon r \geq 0,\ 0 \leq \theta \leq 2\pi,\ 0 \leq \phi \leq \pi\}.$$

The transformation $T: E \to \mathbf{R}^3$ *defined by*

$$T(r, \theta, \phi) = (r\cos\theta\sin\phi, r\sin\theta\sin\phi, r\cos\phi),$$

or as it is usually written

$$x = r\cos\theta\sin\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\phi,$$

is called the **spherical coordinate transformation**, *shown graphically in Figure 7.6.*

a. *Show that* $\lambda(E \setminus E^o) = 0$.

b. *If* $A = \{(x, 0, z) \colon x \geq 0$ *and* $z \in \mathbf{R}\}$, *then show that* A *is a closed subset of* $\mathbf{R}^3$ *whose (3-dimensional) Lebesgue measure is zero.*

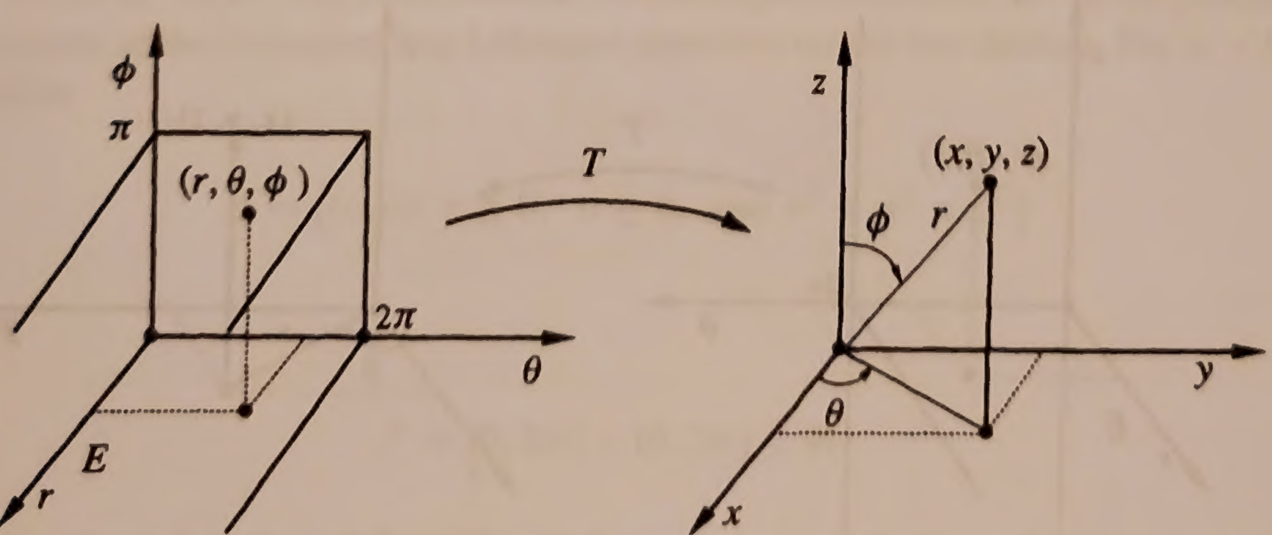

FIGURE 7.6. The Spherical Coordinate Transformation

c. *Show that* $T\colon E^{\circ} \to \mathbf{R}^3 \setminus A$ *is a diffeomorphism whose Jacobian determinant satisfies* $J_T(r,\theta,\phi) = -r^2\sin\phi$.

d. *Show that if* G *is a Lebesgue measurable subset of* E *with* $\lambda(G \setminus G^{\circ}) = 0$, *then* $T(G)$ *is a measurable subset of* $\mathbf{R}^3$. *In addition, show that if* $f \in L_1(T(G))$, *then*

$$\int_{T(G)} f\,d\lambda = \iiint_G f(r\cos\theta\sin\phi, r\sin\theta\sin\phi, r\cos\phi)r^2\sin\phi\,dr\,d\theta\,d\phi$$

holds.

Solution. Repeat the solution of Problem 40.4.

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